TEST 2

201-NYA-05 Calculus 1 for Electronics Engineering Technology
DATE: Thursday April 5th 2012
Instructor: E. Richer
This test is marke out of 50 MARKS

Question 1. *(3 marks each = 6 marks)*
Use implicit differentiation to find the derivatives of the following functions.
(a) $xy^2 + 3x + y = 1$

$$y^2 + 2yy'x + 3 + y' = 0$$
$$2yy'x + y' = -3 - y^2$$
$$y'(2xy + 1) = -3 - y^2$$
$$\boxed{y' = \frac{-3 - y^2}{2xy + 1}}$$

(b) $e^{x \sin y} + x = y^2$

$$e^{x \sin y} \cdot [\sin y + (\cos y) y' x] + 1 = 2yy'$$
$$e^{x \sin y} \cdot \sin y + e^{x \sin y} \cdot \cos y \cdot y' \cdot x + 1 = 2yy'$$
$$e^{x \sin y} \cdot \cos y \cdot y' \cdot x - 2yy' = -1 - \sin y \, e^{x \sin y}$$
$$\boxed{y' = \frac{-1 - (\sin y) e^{x \sin y}}{(x \cos y) e^{x \sin y} - 2y}}$$

Question 2. *(3 marks each = 15 marks)*

Find the derivative.

(a) $f(x) = \tan(x^3 + 7x - 12)$

$$f'(x) = \left[\sec^2(x^3 + 7x - 12)\right] \cdot [3x^2 + 7]$$

(b) $f(x) = \log_3(4x^{-2} + 3)$

$$f'(x) = \left(\frac{1}{4x^{-2}+3}\right) \cdot \left(\frac{1}{\ln 3}\right) \cdot (-8x^{-3})$$

(c) $f(x) = 2^{\sin(2x)}$

$$f'(x) = 2^{\sin 2x} \cdot \ln 2 \cdot \cos 2x \cdot 2$$

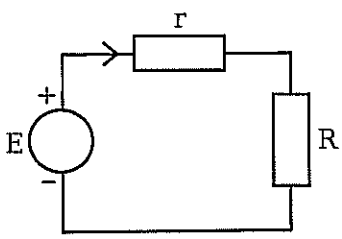
(d) $f(x) = 5^x \cos(\frac{x}{\ln 5})$

$$f'(x) = 5^x \ln 5 \cos\left(\frac{x}{\ln 5}\right) + -\sin\left(\frac{x}{\ln 5}\right) \cdot \left(\frac{1}{\ln 5}\right) 5^x$$

(e) $f(x) = \frac{\log_7(3x+7)}{\tan(3x+7)}$

$$f'(x) = \frac{\left(\frac{1}{3x+7}\right)\left(\frac{1}{\ln 7}\right) \cdot 3 \cdot \tan(3x+7) - \left[\sec^2(3x+7)\right] \cdot 3 \log_7(3x+7)}{\tan^2(3x+7)}$$

Question 3 *(5 marks)*
In the electric circuit shown below, the voltage $E = 5$ (in volts) and resistance $r = 100$ (in ohms) are constant, R is the resistance of a load.



In such a circuit, the electric current i is given by $\frac{E}{r+R}$ and the power P delivered to the load R is given by $P = Ri^2$.

Given that R is positive, determine the value of R so that the power P delivered to R is a maximum.

$$i = \frac{E}{r+R} = \frac{5}{100+R} \qquad ①$$

$$P = Ri^2 \qquad ②$$

SUBSTITUTE ① into ②

$$P = R\left(\frac{5}{100+R}\right)^2 = \frac{25R}{(100+R)^2}$$

$$P' = \frac{25(100+R)^2 - 2(100+R)25R}{(100+R)^4}$$

$$= \frac{25(100+R)[100+R-2R]}{(100+R)^4} = \frac{25[100-R]}{(100+R)^3}$$

CRITICAL PTS $R=100$ & $R=-100$ (but R must be $+$)

INTERVAL	$(0,100)$	$(100,\infty)$
test	1	101
sign P'	$+$	$-$
	$\nearrow$	$\searrow$

$$\boxed{P \text{ IS A MAXIMUM WHEN } R = 100\,\Omega}$$

Question 4. *(5 marks)*
Use logarithmic differentiation to derive the following function $y = (\cos x)^x$

$$\ln y = \ln(\cos x)^x$$
$$\ln y = x \ln \cos x$$
$$\frac{1}{y} y' = \ln \cos x + \frac{1}{\cos x}(-\sin x) \cdot x$$
$$y' = y\left[\ln \cos x - (\tan x) x\right]$$
$$\boxed{y' = [\cos x]^x \left[\ln \cos x - x \tan x\right]}$$

Question 5. *(5 marks)*
Determine where the function $f(x) = x^3 - 6x^2 - 12x + 2$ is concave up and identify any of its inflection points.

$$f'(x) = 3x^2 - 12x - 12$$
$$f''(x) = 6x - 12$$
$$= 6(x-2)$$

critical point at $x = 2$

Interval	$(-\infty, 2)$	$(2, \infty)$
Test	0	3
sign of f''	$-$	$+$
	$\cap$	$\cup$

$f(x)$ is concave up on the interval $(2, \infty)$

There is an inflexion point at $x = 2$

$$y = f(2) = 2^3 - 6(2)^2 - 12(2) + 2$$
$$= 8 - 24 - 24 + 2$$
$$= -38$$

$$(x, y) = (2, -38)$$

Question 6. *(5 marks)*
Determine where the function $f(x) = 2x^3 - 6x^2 + 6x - 5$ is decreasing.

$$f'(x) = 6x^2 - 12x + 6$$
$$= 6(x^2 - 2x + 1)$$
$$= 6(x-1)^2$$

Critical pt @ $x = 1$

Interval	$(-\infty, 1)$	$(1, \infty)$
test pt	0	2
Sign f'	+	+
	↗	↗

$\boxed{f(x) \text{ is never decreasing}}$

Question 7. *(4 marks)*
Find all the local extrema of the function $(x+2)^5(x-3)^4$.

$$f'(x) = 5(x+2)^4(x-3)^4 + 4(x-3)^3(x+2)^5$$
$$= (x+2)^4(x-3)^3\left[5(x-3) + 4(x+2)\right]$$
$$= (x+2)^4(x-3)^3\left[5x - 15 + 4x + 8\right]$$
$$= (x+2)^4(x-3)^3\left[9x - 7\right]$$

Critical points $x = -2, 3$ & $7/9$

Intervals	$(-\infty, -2)$	$(-2, 7/9)$	$(7/9, 3)$	$(3, \infty)$
Test pt	-3	0	1	4
Sign of f'	+	+	—	+
	↗	↗	↘	↗

MAX @ $x = 7/9$ $y =$

Min @ $x = 3$ $y = 0$

Question 8. *(2.5 marks each = 5 marks)*

Find the **second** derivative of each of the following functions:

(a) $f(x) = x\tan x$

$$f'(x) = \tan x + x\sec^2 x$$
$$= \tan x + x[\cos x]^{-2}$$

$$f''(x) = \sec^2 x + (\cos x)^{-2} - 2[\cos x]^{-3}(-\sin x)x$$
$$= \sec^2 x + \sec^2 x + \frac{2x\sin x}{\cos^3 x}$$
$$= \sec^2 x + \sec^2 x + 2x\tan x\sec^2 x$$
$$= \sec^2 x\,[2 + 2x\tan x]$$
$$= 2\sec^2 x\,[1 + x\tan x]$$

(b) $f(x) = \log_2(x^2+1)$

$$f'(x) = \left(\frac{1}{x^2+1}\right)\left(\frac{1}{\ln 2}\right)2x$$
$$= \left(\frac{1}{\ln 2}\right)\left(\frac{2x}{x^2+1}\right)$$

$$f''(x) = \frac{1}{\ln 2}\left[\frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}\right]$$
$$= \frac{1}{\ln 2}\left[\frac{-2x^2+2}{(x^2+1)^2}\right]$$

Bonus. *(5 marks)*
Find the derivative of the function $y = [\ln(x^x)]^x$

$$\ln y = \ln [\ln(x^x)]^x$$

$$\ln y = x \ln [x \ln x]$$

$$\frac{1}{y} y' = \ln(x \ln x] + \frac{1}{x \ln x} [\ln x + \frac{1}{x} \cdot x] \cdot x$$

$$y' = y \left[\ln(x \ln x] + 1 + \frac{1}{\ln x} \right]$$

$$y' = [\ln(x^x)]^x \left[\ln(x \ln x] + 1 + \frac{1}{\ln x} \right]$$