# Partial Fractions

In this section we will examine a method of integration for frational functions.

Rational Function: $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

**Common Types of Integrals**

We will need to be able to integrate the following integrals the following integrals.

**1)** $\displaystyle\int \frac{1}{(ax+b)^n}\,dx$

For this type of integral we use a substitution.

Examples: Find the following antiderivatives.

**a)** $\displaystyle\int \frac{1}{(3x+2)}\,dx$

**b)** $\displaystyle\int \frac{4}{(5x-7)^3}\,dx$

**2)** $\int \frac{ax+b}{(cx^2+dx+g)^n}\,dx$

We want to try to make the numberator look like the derivative of the denominator so we can use a substitution.

Examples: Find the following antiderivatives.

**a)** $\int \frac{8x+3}{4x^2+3x-4}\,dx$

**b)** $\int \frac{2x+1}{(2x^2+2x+5)^3}\,dx$

**b)** $\int \frac{5x-7}{2x^2+2}\,dx$

Remeber: the goal of this section is to integrate rational functions, $p(x)/q(x)$. If the degree of the numberator $p(x)$ is **greater than** or **equal to** the degree of the denominator $q(x)$ we first perform **long division**.
Example: Divide the following using long division.

**a)** $\dfrac{3x^2+4x-3}{x+2}$

**b)** $\dfrac{3x^3-5x+2}{x-1}$

**c)** $\dfrac{4x^3 + 8x^2 - x + 6}{x^2 + 1}$

Example: Find the following.

**1)** $\displaystyle\int \frac{x^3+3x^2}{x^2+1}\,dx$

**2)** $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

Partial Fractions: The next step is to factor the denominator as much as possible. It can be shown that any polynomial $q(x)$ can be factored as a product of linear factors (of the form $ax+b$) and irreducible quadratic factors (of the form $ax^2+bx+c$ where $b^2-4ac<0$).

Important: All remaining quadratis must be irreducible!!!

Once $q(x)$ is factored the next step is to express the function as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

To illustrate, recall that we can combine fractions

$$\frac{5}{x+3}-\frac{2}{x+1}$$

and so we can separate

**Case 1**: The denominator $q(x)$ is a product of distinct linear factors $ax+b$.

$$q(x) = (a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)$$

For each linear factor we need one term of the type $\frac{A}{ax+b}$.

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_n}{a_nx+b_n}$$

Example: Write the following rational function as a sum or simple fractions

$$\frac{x+5}{(x-4)(x-1)}$$

Example: Find

1) $\displaystyle\int \frac{x+5}{x^2-5x+4}\,dx$

2) $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

3) $\int \frac{2x-2}{(x+5)(x+2)(x-3)}\,dx$

**Case 2**: The denominator $q(x)$ is a product of linear factors $ax+b$ some of which are repeated. For each $(ax+b)^n$ we need

$$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots+\frac{A_n}{(ax+b)^n}$$

For example if we wanted to write the following as a sum of simpler fractions we would need

$$\frac{x^2+2x-3}{x^2(2x+1)^3}$$

Example: Find

1) $\displaystyle\int \frac{x^2-2x-5}{x^3-5x^2}\,dx$

2) $\int \frac{5x^2 - 9x}{(x-4)(x-1)^2}\,dx$

3) $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

**Case 3**: The denominator $q(x)$ contains irreducible quadratic factors $ax^2+bx+c$, where $b^2-4ac<0$, none of which are repeated. For each $ax^2+bx+c$ we need a factor of the type

$$\frac{Ax+B}{ax^2+bx+c}$$

For example if we wanted to write the following as a sum of simpler fractions we would need

$$\frac{x^2+2x}{(x+1)^3(x^2+x+1)(3x^2+2)}$$

Example: Find

1) $\displaystyle\int \frac{4x^2+9x+2}{x^3+3x^2+x}\,dx$

2) $\int \frac{-5x^3+2x^2+x+3}{x^4+x^2}\,dx$