Partial Fractions

In this section we will examine a method of integration for ~~frational~~ rational functions.

Rational Function: $\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Ex: $\displaystyle\int \frac{x^4-3x^3+2x-1}{x^2-3x+2}\,dx$

Common Types of Integrals

We will need to be able to integrate the following integrals the following integrals.

1) $\displaystyle\int \frac{1}{(ax+b)^n}\,dx$

For this type of integral we use a substitution.

Examples: Find the following antiderivatives.

a) $\displaystyle\int \frac{1}{(3x+2)}\,dx = \frac{\ln|3x+2|}{3} + C$

$u = 3x+2$

$du = 3dx$

b) $\displaystyle\int \frac{4}{(5x-7)^3}\,dx = 4\int \frac{1}{(5x-7)^3}\,dx = 4\int u^{-3}\,\frac{du}{5}$

$u = 5x-7$

$du = 5dx$

$\displaystyle = \frac{4}{5}\,\frac{u^{-2}}{-2} + C = -\frac{4}{10}\cdot\frac{1}{(5x-7)^2} + C$

2) $\displaystyle\int \frac{ax+b}{(cx^2+dx+g)^n}\,dx$

We want to try to make the numberator look like the derivative of the denominator so we can use a substitution.

Examples: Find the following antiderivatives.

a) $\displaystyle\int \frac{8x+3}{4x^2+3x-4}\,dx$

LET $u = 4x^2+3x-4$

$du = (8x+3)dx$

$\dfrac{du}{8x+3} = dx$

$$= \int \frac{8x+3}{u}\cdot\frac{du}{8x+3} = \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \ln|4x^2+3x-4| + C$$

b) $\displaystyle\int \frac{2x+1}{(2x^2+2x+5)^3}\,dx$

LET $u = 2x^2+2x+5$

$du = (4x+2)dx$

$\dfrac{du}{2(2x+1)} = dx$

$$= \int \frac{\cancel{2x+1}}{u^3}\,\frac{du}{2(\cancel{2x+1})} = \frac{1}{2}\int u^{-3}\,du$$

$$= \frac{1}{2}\frac{u^{-2}}{-2} + C = -\frac{1}{4}\cdot\frac{1}{(2x^2+2x+5)^2} + C$$

c) $\displaystyle\int \frac{5x-7}{2x^2+2}\,dx = \int \frac{5x}{2x^2+2}\,dx - \int \frac{7}{2x^2+2}\,dx$

LET $u = 2x^2+2$

$du = 4x\,dx$

$\dfrac{du}{4x} = dx$

$$= \int \frac{5x}{u}\cdot\frac{du}{4x} - \int \frac{7}{2x^2+2}\,dx = \frac{5}{4}\ln|2x^2+2| - \int \frac{7}{2x^2+2}\,dx$$

$$= \frac{5}{4}\ln|2x^2+2| - \frac{7}{2}\int \frac{1}{x^2+1}\,dx = \frac{5}{4}\ln|2x^2+2| - \frac{7}{2}\arctan x + C$$

Remeber: the goal of this section is to integrate rational functions, $p(x)/q(x)$. If the degree of the numberator $p(x)$ is **greater than** or **equal to** the degree of the denominator $q(x)$ we first perform **long division**.
Example: Divide the following using long division.

a) $\dfrac{3x^2+4x-3}{x+2}$

$$\frac{17}{7} = 2 + \frac{3}{7} \leftarrow \text{REMAINDER}$$

$$\frac{11}{2} = 5 + \frac{1}{2}$$

$$\begin{array}{r l}
 & 3x - 2 \\
x+2 \,\big) & 3x^2 + 4x - 3 \\
3x(x+2) \rightarrow & -(3x^2 + 6x) \\
 & \quad -2x - 3 \\
-2(x+2) \rightarrow & \quad -(-2x - 4) \\
 & \qquad\qquad 1
\end{array}$$

1 ← REMAINDER
(DEGREE IS LESS THAN
THE DENOMINATOR)

$$\frac{3x^2+4x-3}{x+2} = 3x-2 + \frac{1}{x+2}$$

b) $\dfrac{3x^3-5x+2}{x-1}$

$$\begin{array}{r l}
 & 3x^2 + 3x - 2 \\
x-1 \,\big) & 3x^3 + 0x^2 - 5x + 2 \\
 & -(3x^3 - 3x^2) \\
 & \qquad 3x^2 - 5x \\
 & \quad -(3x^2 - 3x) \\
 & \qquad\quad -2x + 2 \\
 & \quad -(-2x + 2) \\
 & \qquad\qquad 0
\end{array}$$

$$\frac{3x^3-5x+2}{x-1}$$

$$= 3x^2+3x-2 + \frac{0}{x-1}$$

$$= 3x^2+3x-2$$

c) $\dfrac{4x^3+8x^2-x+6}{x^2+1}$

$$
\begin{array}{r}
4x+8 \\
x^2+0x+1 \overline{\big) \, 4x^3+8x^2-x+6} \\
-(4x^3+0x^2+4x) \\
\hline
8x^2-5x+6 \\
-(8x^2+0x+8) \\
\hline
-5x-2
\end{array}
$$

$$\frac{4x^3+8x^2-x+6}{x^2+1} = 4x+8 - \frac{5x+2}{x^2+1}$$

Example: Find the following.

1) $\displaystyle\int \frac{x^3+3x^2}{x^2+1}\,dx = I$

$$\begin{array}{r|l}
 & x \;+3 \\ \hline
x^2+0x+1 & x^3+3x^2+0x+0 \\
 & -(x^3+0x^2+x) \\ \cline{2-2}
 & \qquad 3x^2-x+0 \\
 & \quad -(3x^2+0x+3) \\ \cline{2-2}
 & \qquad\quad -x-3
\end{array}$$

$$I = \int x+3 - \frac{x+3}{x^2+1}\,dx = \int x+3\,dx - \int \frac{x}{\underbrace{x^2+1}_{\substack{u=x^2+1 \\ du=2x\,dx}}}\,dx - \int \frac{3}{x^2+1}\,dx$$

$$= \frac{x^2}{2}+3x - \int \frac{\cancel{x}}{u}\frac{du}{2\cancel{x}} - 3\int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^2}{2}+3x - \frac{1}{2}\ln|u| - 3\arctan x + C$$

$$= \frac{x^2}{2}+3x - \frac{1}{2}\ln|x^2+1| - 3\arctan x + C$$

2) $\displaystyle\int \frac{x^3+x}{x-1}\,dx = I$

$$\begin{array}{r}
x^2 + x + 2 \\
x-1 \;\overline{)\; x^3 + 0x^2 + x + 0} \\
-(x^3 - x^2) \qquad\qquad \\
\hline
x^2 + x \qquad \\
-(x^2 - x) \qquad \\
\hline
2x + 0 \\
-(2x - 2) \\
\hline
2
\end{array}$$

$$I = \int x^2 + x + 2 + \frac{2}{x-1}\,dx = \int x^2+x+2\,dx + 2\int \frac{1}{x-1}\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

ONCE THE DEGREE OF THE NUMERATOR IS LESS THAN THE DENOMINATOR

Partial Fractions: The next step ✓ is to factor the denominator as much as possible. It can be shown that any polynomial $q(x)$ can be factored as a product of linear factors (of the form $ax+b$) and irreducible quadratic factors (of the form ax^2+bx+c where $b^2-4ac<0$).

$q(x)$

FACTOR

FACTORS OF THE FORM $ax+b$

FACTORS OF THE FORM ax^2+bx+c WHERE $b^2-4ac<0$

Important: All remaining quadratics must be irreducible!!!

Once $q(x)$ is factored the next step is to express the function as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^j}$$

To illustrate, recall that we can combine fractions

$$\frac{5}{x+3}-\frac{2}{x+1} = \frac{5(x+1)}{(x+3)(x+1)} - \frac{2(x+3)}{(x+3)(x+1)}$$

$$= \frac{(5x+5)-(2x+6)}{(x+3)(x+1)} = \frac{3x-1}{x^2+4x+3}$$

and so we can separate

$$\frac{3x-1}{x^2+4x+3} = \frac{5}{x+3} - \frac{2}{x+1}$$

HOW DO WE DO THIS?

THERE ARE FOUR CASES THAT WE WILL LOOK AT

Case 1: The denominator $q(x)$ is a product of distinct linear factors $ax+b$.

$$q(x) = (a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)$$

Ex: $q(x) = (2x+1)(3x-7)(5x+2)\dots(6x-1)$

For each linear factor we need one term of the type $\frac{A}{ax+b}$.

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_n}{a_nx+b_n}$$

Example: Write the following rational function as a sum or simple fractions

$$\frac{x+5}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1}$$

← HOW DO WE FIND THESE CONSTANTS?

$$\cancel{(x-4)}\cancel{(x-1)} \cdot \frac{x+5}{\cancel{(x-4)}\cancel{(x-1)}} = \cancel{(x-4)}(x-1)\frac{A}{\cancel{x-4}} + (x-4)\cancel{(x-1)}\frac{B}{\cancel{x-1}}$$

MULTIPLY BOTH SIDES BY $(x-4)(x-1)$.

THIS EQUALITY IS TRUE FOR ALL x VALUES.

$$x+5 = (x-1)A + (x-4)B$$

IF $\underline{x=1}$

$$1+5 = (1-1)A + (1-4)B$$

$$6 = 0A - 3B$$

$$\underline{-2 = B}$$

$\underline{\text{IF } x=4}$

$$4+5 = (4-1)A + (4-4)(-2)$$

$$9 = 3A + 0$$

$$3 = A$$

$$\therefore \frac{x+5}{(x-4)(x-1)} = \frac{3}{x-4} + \frac{(-2)}{x-1}$$

Example: Find

1) $\int \frac{x+5}{x^2-5x+4}\,dx = \int \frac{x+5}{(x-4)(x-1)}\,dx$

$$= \int \frac{3}{x-4} - \frac{2}{x-1}\,dx = 3\int \frac{1}{x-4}\,dx - 2\int \frac{1}{x-1}\,dx$$

$$= 3\ln|x-4| - 2\ln|x-1| + C$$

2) $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

3) $\displaystyle\int \frac{2x-2}{(x+5)(x+2)(x-3)}\,dx$

Case 2: The denominator $q(x)$ is a product of linear factors $ax+b$ some of which are repeated. For each $(ax+b)^n$ we need

$$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots+\frac{A_n}{(ax+b)^n}$$

For example if we wanted to write the following as a sum of simpler fractions we would need

$$\frac{x^2+2x-3}{x^2(2x+1)^3}$$

Example: Find

1) $\displaystyle\int \frac{x^2-2x-5}{x^3-5x^2}\,dx$

2) $\displaystyle\int \frac{5x^2 - 9x}{(x-4)(x-1)^2}\,dx$

3) $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

Case 3: The denominator $q(x)$ contains irreducible quadratic factors ax^2+bx+c, where $b^2-4ac<0$, none of which are repeated. For each ax^2+bx+c we need a factor of the type

$$\frac{Ax+B}{ax^2+bx+c}$$

For example if we wanted to write the following as a sum of simpler fractions we would need

$$\frac{x^2+2x}{(x+1)^3(x^2+x+1)(3x^2+2)}$$

Example: Find

1) $\displaystyle\int \frac{4x^2+9x+2}{x^3+3x^2+x}\,dx$

2) $\displaystyle\int \frac{-5x^3+2x^2+x+3}{x^4+x^2}\,dx$