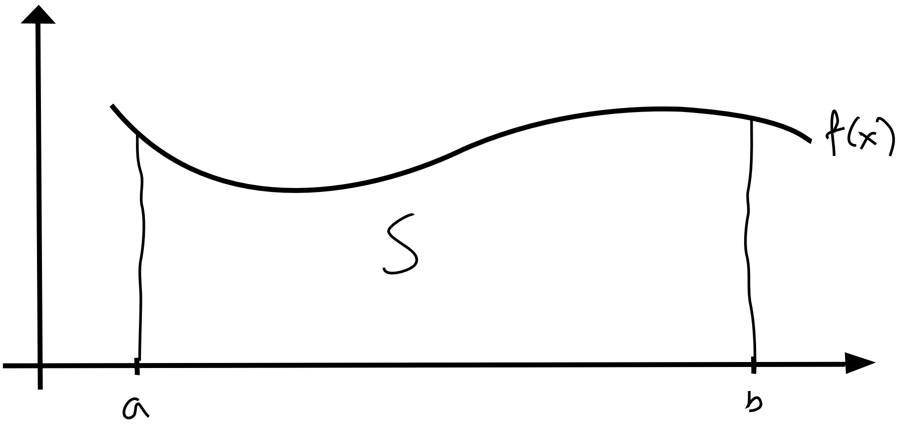
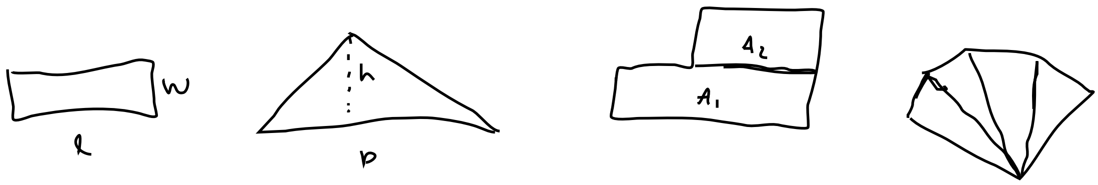
The Area Problem

Suppose we want to find the area of the region S that lies under the graph of $y = f(x)$ from $x = a$ to $x = b$.



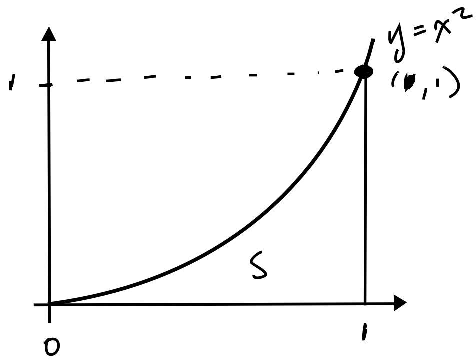
To find the area of a polygon, we can divide it into smaller shapes and find the area of them:



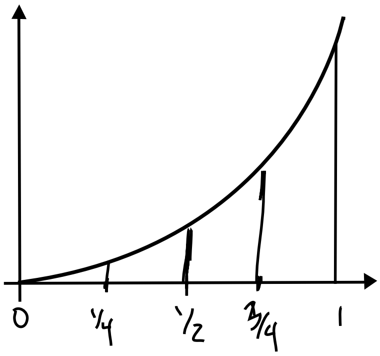
$A = l \cdot w$ $\qquad$ $A = \dfrac{b \cdot h}{2}$ $\qquad$ $A = A_1 + A_2$ $\qquad$ $A = A_1 + A_2 + A_3 + A_4$

Unfortunately this method doesn't seem to work for a region with curved sides. Part of the area problem is coming up with a precise definition of this area.

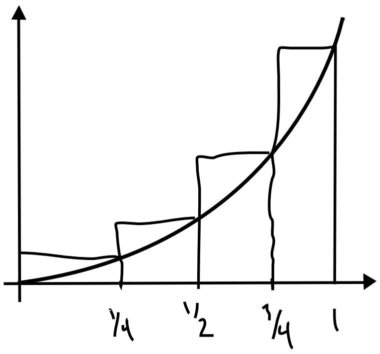
Let's start by trying to estimate the area under $y = x^2$ from 0 to 1.



We know that S must be somewhere between 0 and 1 since it is contained in a square with side length 1.
Suppose we divide S into four strips at $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$ and $x_4 = 1$.

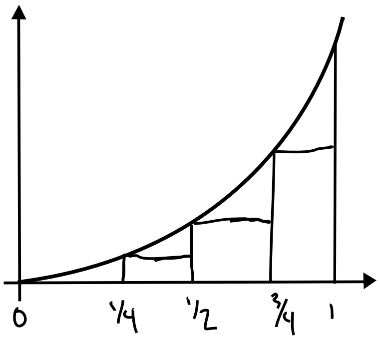


We can approximate the area of each strip by using rectangles.



$$R_4 = \frac{1}{4} f(1/4) + \frac{1}{4} f(1/2) + \frac{1}{4} f(3/4) + \frac{1}{4} f(1)$$
$$= \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}(1/2)^2 + \frac{1}{4}(3/4)^2 + \frac{1}{4}(1)^2$$
$$= 0.46875$$

So if A is the area of S we see that $A < R_4 = 0.46875$. Noticed that for our approximation we used the values of f at the **right endpoints** of each subinterval. If we instead use the values of f at the **left endpoints** we get



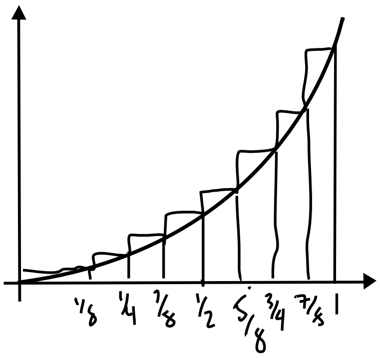
$$L_4 = \frac{1}{4} f(0) + \frac{1}{4} f(1/4) + \frac{1}{4} f(1/2) + 1/4 f(3/4)$$
$$= \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}(1/2)^2 + 1/4\,(3/4)^2$$
$$= 0.21875$$

The area A is larger than $\cancel{R_4}$ L_4 so

$$L_4 < A < R_4$$
$$0.21875 < A < 0.46875$$

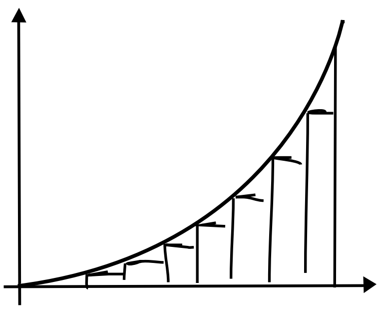
If we use more rectangles our approximation gets better.

Eight Strips



$$R_8 = \frac{1}{8}\left(\frac{1}{8}\right)^2 + \frac{1}{8}\left(\frac{1}{4}\right)^2 + \frac{1}{8}\left(\frac{3}{8}\right)^2 + \frac{1}{8}\left(\frac{1}{2}\right)^2 + \frac{1}{8}\left(\frac{5}{8}\right)^2 + \frac{1}{8}\left(\frac{3}{4}\right)^2 + \frac{1}{8}\left(\frac{7}{8}\right)^2 + \frac{1}{8}\left(1\right)^2$$

$$= 0.3984375$$



$$L_8 = \frac{1}{8}(0)^2 + \frac{1}{8}\left(\frac{1}{8}\right)^2 + \frac{1}{8}\left(\frac{1}{4}\right)^2 + \frac{1}{8}\left(\frac{3}{8}\right)^2 + \frac{1}{8}\left(\frac{1}{2}\right)^2 + \frac{1}{8}\left(\frac{5}{8}\right)^2 + \frac{1}{8}\left(\frac{3}{4}\right)^2 + \frac{1}{8}\left(\frac{7}{8}\right)^2 = 0.2734375$$

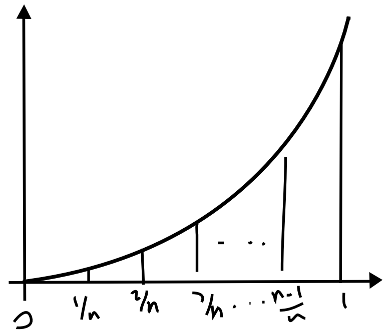
$$0.2734375 < A < 0.3984375$$

We see we get better and better estimates by increasing the number of strips. The following table shows the results of using n strips.

n	L_n	R_n
10	0.285000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.323400	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

It looks like R_n and L_n ~~is~~ are approaching $\frac{1}{3}$ as n increases.

Using *n* rectangles the approximation becomes



EACH RECTANGLE HAS WIDTH $1/n$

WITH ENDPOINTS

$0, 1/n, 2/n, 3/n, \ldots, \frac{n-1}{n}, 1$

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \ldots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n}\left[\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \ldots + \frac{n^2}{n^2}\right]$$

$$= \frac{1}{n^3}\left[1^2+2^2+3^2+\ldots+n^2\right] = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

THE MORE RECTANGLES WE USE THE BETTER OUR APPROXIMATION WILL GET

IF WE LET $n \to \infty$ WE SHOULD GET THE EXACT AREA

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{\cancel{n^3}\, n^2}\,\frac{\cancel{n}(n+1)(2n+1)}{6} = \lim_{n\to\infty} \frac{2n^2+2n+n+1}{6n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}}{\frac{6n^2}{n^2}} = \lim_{n\to\infty} \frac{\overset{\nearrow 2}{2} + \overset{\nearrow 0}{3/n} + \overset{\nearrow 0}{1/n^2}}{\underset{\searrow 6}{6}} = \frac{2+0+0}{6} = \frac{1}{3}$$

IT IS ALSO THE CASE THAT $\lim_{n\to\infty} L_n = 1/3$

YOU TRY:

LET $f(x) = 4-2x$

a) SKETCH THE REGION R UNDER THE GRAPH OF f ON THE INTERVAL $[0,2]$ AND FIND ITS EXACT AREA USING GEOMETRY.

b) USE A RIEMANN SUM (A SUM OF APPROXIMATING RECTANGLES) OF EQUAL WIDTH $(n=5)$ TO APPROXIMATE THE AREA OF R. USE BOTH THE LEFT ENDPOINTS THEN THE RIGHT ENDPOINTS

c) REPEAT b) WITH $n=10$.