The Riemann Sum and the Definite Integral

Let's apply this idea to use approximating rectangles fing the area of the region S under a positive function $f(x)$ on the interval $[a,b]$.

The width of the interval $[a,b]$ is $b-a$ so the width of each strip is

$$\Delta x =$$

The strips divide the interval into n subintervals $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, $\ldots [x_{n-1},x_n]$. The endpoints are

Using the right endpoints we approximate the area with

$$R_n =$$

This sum of the areas of approximating rectangles is called a **Riemann Sum**.
As we saw last class, the approximation gets better as n increases. Because this is the case we define the following:

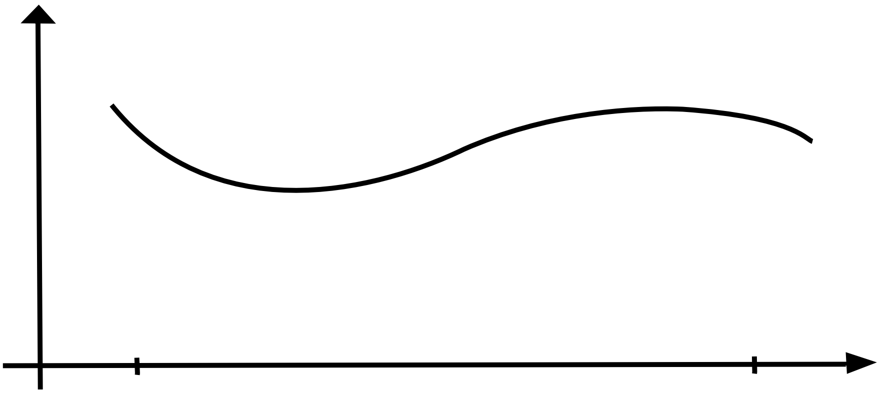
Definition: The **Area** or the region S that lies under the graph of a positive continuous function f is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x]$$

It can be shownthat this limit always exists (for a continuous function) and we get the same value if we use

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_{n-1})\Delta x]$$

In fact, instead of using the left or the right endpoints for the heights of the rectangles we can pick any point we want x_i* from each of the subintervals $[x_{i-1}, x_i]$.



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Notice that we can write this definition using sigma notation:

$$\begin{aligned} A = \lim_{n\to\infty} R_n &= \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \end{aligned}$$

This limit is called the **definite integral**.

Definition: If $f(x)$ is a continuous function defined on $[a, b]$, and if $[a, b]$ is divided into n equal subintervals of width $\Delta x = \frac{b-a}{n}$, and if $x_i = a + i\Delta x$ is the right endpoint of the subinterval $[x_{i-1}, x_i]$ then the **definite integral** of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Example: Use the definition of the definite integral to evaluate the following

1) $\int_0^3 (x - 2x^2)\, dx$

2) $\int_{2}^{5} (8x - x^2)\, dx$

3) $\int_0^2 (x^2 + 10)\, dx$

4) $\int_{1}^{5} (x - 4x^2)\, dx$

5) $\int_{-3}^{0} (4x^2 - 5x - 1)\, dx$

6) $\int_0^4 (x^3 - 1)\, dx$