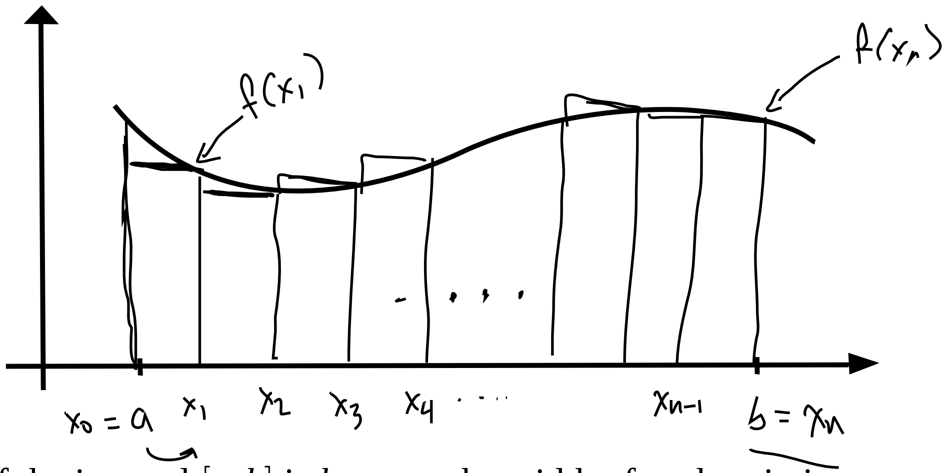
The Riemann Sum and the Definite Integral

Let's apply this idea to use approximating rectangles fing the area of the region S under a positive function $f(x)$ on the interval $[a,b]$.



The width of the interval $[a,b]$ is $b-a$ so the width of each strip is

$$\Delta x = \frac{b-a}{n}$$

The strips divide the interval into n subintervals $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, $\ldots [x_{n-1},x_n]$. The endpoints are

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x$$

$$\ldots \quad x_i = a + i\Delta x, \quad x_n = a + n\Delta x = b$$

Using the right endpoints we approximate the area with

$$R_n = \Delta x f(x_1) + \Delta x f(x_2) + \ldots + \Delta x f(x_n)$$

This sum of the areas of approximating rectangles is called a **Riemann Sum**.
As we saw last class, the approximation gets better as n increases. Because this is the case we define the following:

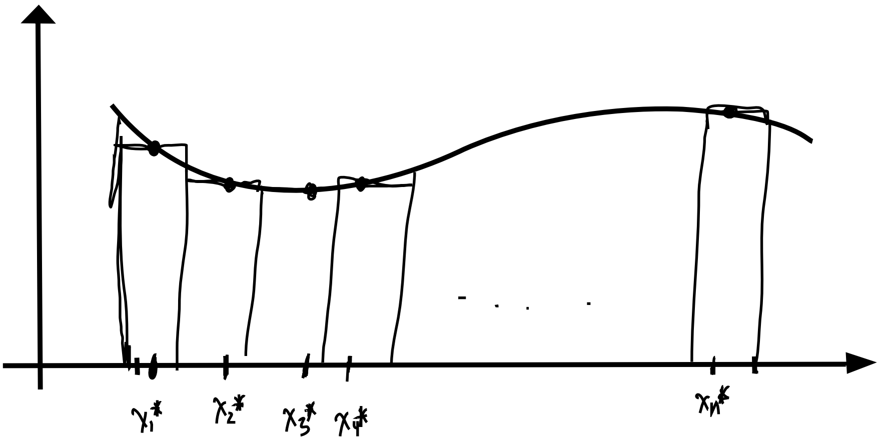
Definition: The **Area** or the region S that lies under the graph of a positive continuous function f is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x]$$

It can be shownthat this limit always exists (for a continuous function) and we get the same value if we use

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_{n-1})\Delta x]$$

In fact, instead of using the left or the right endpoints for the heights of the rectangles we can pick any point we want x_i* from each of the subintervals $[x_{i-1}, x_i]$.



Still

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x]$$

Notice that we can write this definition using sigma notation:

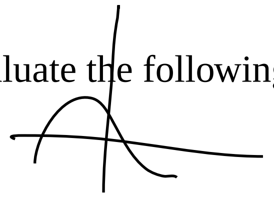
$$\begin{aligned} A = \lim_{n\to\infty} R_n &= \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x] \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \end{aligned}$$

This limit is called the **definite integral**.

Definition: If $f(x)$ is a continuous function defined on $[a,b]$, and if $[a,b]$ is divided into n equal subintervals of width $\Delta x = \frac{b-a}{n}$, and if $x_i = a + i\Delta x$ is the right endpoint of the subinterval $[x_{i-1}, x_i]$ then the **definite integral** of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Example: Use the definition of the definite integral to evaluate the following



1) $\int_0^3 (x-2x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

- $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{n} = \dfrac{3}{n}$ • $x_i = a + i\Delta x = 0 + i\left(\dfrac{3}{n}\right) = \dfrac{3i}{n}$

- $f(x_i) = x_i - 2(x_i)^2 = \dfrac{3i}{n} - 2\left(\dfrac{3i}{n}\right)^2 = \dfrac{3i}{n} - 2\cdot\dfrac{9i^2}{n^2} = \dfrac{3i}{n} - \dfrac{18i^2}{n^2}$

- $f(x_i)\Delta x = \underbrace{\left[\dfrac{3i}{n} - \dfrac{18i^2}{n^2}\right]}_{f(x_i)} \cdot \underbrace{\dfrac{3}{n}}_{\Delta x} = \dfrac{9i}{n^2} - \dfrac{54i^2}{n^3}$

- $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[\frac{9i}{n^2} - \frac{54i^2}{n^3}\right] = \sum_{i=1}^{n}\frac{9i}{n^2} - \sum_{i=1}^{n}\frac{54i^2}{n^3}$

$$= 9\sum_{i=1}^{n}\frac{i}{n^2} - 54\sum_{i=1}^{n}\frac{i^2}{n^3} = \frac{9}{n^2}\sum_{i=1}^{n} i - \frac{54}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{9}{n^2}\cdot\frac{n(n+1)}{2} - \frac{54}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

- $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[\frac{9}{n^2}\cdot\frac{n(n+1)}{2} - \frac{54}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$

$$= \lim_{n\to\infty}\frac{9}{2}\cdot\frac{n+1}{n} - \lim_{n\to\infty} 9\cdot\frac{(n+1)(2n+1)}{n^2} = \frac{9}{2}\lim_{n\to\infty}\frac{1+1/n}{1} - 9\lim_{n\to\infty}\frac{2n^2+3n+1}{n^2}$$

$$= \frac{9}{2}\lim_{n\to\infty}\frac{1+1/n}{1} - 9\lim_{n\to\infty}\frac{2+3/n+1/n^2}{1} = \frac{9}{2}\cdot\frac{1+0}{1} - 9\,\frac{2+0+0}{1}$$

$$= \frac{9}{2} - 18 = \frac{-27}{2}$$

$$\therefore \int_0^3 (x-2x^2)\,dx = -\frac{27}{2}$$

2) $\displaystyle\int_2^5 (8x - x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

• $\Delta x = \dfrac{b-a}{n} = \dfrac{5-2}{n} = \dfrac{3}{n}$ • $x_i = a + i\Delta x = 2 + i\left(\frac{3}{n}\right) = 2 + \dfrac{3i}{n}$

• $f(x_i) = 8x_i - (x_i)^2 = 8\left(2 + \dfrac{3i}{n}\right) - \left(2 + \dfrac{3i}{n}\right)^2$ ← $\left(2+\frac{3i}{n}\right)\left(2+\frac{3i}{n}\right)$

$= 16 + \dfrac{24i}{n} - \left(4 + \dfrac{12i}{n} + \dfrac{9i^2}{n^2}\right)$

$= 12 + \dfrac{12i}{n} - \dfrac{9i^2}{n^2}$

MAKE SURE YOU USE Δx NOT x_i!

• $\underline{f(x_i)\Delta x} = \left[12 + \dfrac{12i}{n} - \dfrac{9i^2}{n^2}\right] \cdot \dfrac{3}{n}$

$= \dfrac{36}{n} + \dfrac{36i}{n^2} - \dfrac{27i^2}{n^3}$

• $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[\frac{36}{n} + \frac{36i}{n^2} - \frac{27i^2}{n^3}\right]$

$\displaystyle = \sum_{i=1}^{n}\frac{36}{n} + \sum_{i=1}^{n}\frac{36i}{n^2} - \sum_{i=1}^{n}\frac{27i^2}{n^3} = \frac{36}{n}\sum_{i=1}^{n} 1 + \frac{36}{n^2}\sum_{i=1}^{n} i - \frac{27}{n^3}\sum_{i=1}^{n} i^2$

$= \dfrac{36}{n}\cdot n + \dfrac{36}{n^2}\cdot\dfrac{n(n+1)}{2} - \dfrac{27}{n^3}\cdot\dfrac{n(n+1)(2n+1)}{6}$

• $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[36 + 18\cdot\frac{n+1}{n} - \frac{9}{2}\cdot\frac{2n^2+3n+1}{n^2}\right]$

$\displaystyle = \lim_{n\to\infty} 36 + 18\lim_{n\to\infty}\frac{1+1/n}{1} - \frac{9}{2}\lim_{n\to\infty}\frac{2+3/n+1/n^2}{1}$

$= 36 + 18(1) - \frac{9}{2}(2) = 45$

$\therefore \displaystyle\int_2^5 (8x - x^2)\,dx = 45$

3) $\displaystyle\int_0^2 (x^2+10)\,dx = \frac{68}{3}$

- $\Delta x = \frac{2}{n}$ • $x_i = \frac{2i}{n}$

- $f(x_i) = \frac{4i^2}{n^2} + 10$ • $f(x_i)\Delta x = \frac{8i^2}{n^3} + \frac{20}{n}$

- $$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[\frac{8i^2}{n^3} + \frac{20}{n}\right] = \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{20}{n}\sum_{i=1}^{n} 1$$

$$= \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{20}{n}\cdot n$$

- $$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{4}{3}\lim_{n\to\infty}\frac{2+3/n+1/n^2}{1} + \lim_{n\to\infty} 20$$

$$= \frac{8}{3} + 20 = \frac{68}{3}$$

4) $\int_{1}^{5} (x - 4x^2)\, dx$ $= -\dfrac{460}{3}$

5) $\int_{-3}^{0} (4x^2 - 5x - 1)\, dx$ $= \frac{111}{2}$

6) $\int_0^4 (x^3 - 1)\, dx$ $= 60$