Dawson College: **Calculus I**: 201-NYA-05-S4: Winter 2012

Name: Y. Lamontagne
Student ID:

Test 3

This test is graded out of 60 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Consider the function

$$f(x) = \frac{-1}{3}x^3 + x^2 + 3x$$

a. *(1 mark)* State the domain of $f(x)$.
b. *(3 marks)* Find the x-intercepts (if any) and the y-intercept.
c. *(3 marks)* Find the intervals where $f(x)$ is increasing, and the intervals where $f(x)$ is decreasing.
d. *(1 mark)* Find the local maxima and local minima.
e. *(3 marks)* Find the intervals where $f(x)$ is concave up, and the intervals where $f(x)$ is concave down. Give the inflection point(s).
f. *(4 marks)* Use the information above to sketch the graph of $f(x)$ *(your graph has to agree with the previous answers)*. Clearly indicate the coordinates of the relative extremum(s), the intercept(s), the inflection point(s), if any.

a. Domain: $\mathbb{R}$

b. y-int: $(0, f(0)) = (0,0)$

x-int: $0 = f(x)$

$0 = \frac{-1}{3}x^3 + x^2 + 3x$

$0 = x(x^2 - 3x - 9)$

$x = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-9)}}{2(1)}$

$= \frac{3 \pm \sqrt{45}}{2} \doteq 4.85, -1.85$

c. critical points: $f'(x) = -x^2 + 2x + 3$

So $0 = f'(x)$

$0 = -x^2 + 2x + 3$

$0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$x = 3$ $\quad x = -1$

	$(-\infty, -1)$	$(-1, 3)$	$(3, \infty)$
p	-2	0	4
$f'(p)$	$f'(-2) = -5$ $-$	$f'(0) = 3$ $+$	$f''(4) = -5$ $-$
inc./dec	↘	↗	↘

d. ∴ at $x = -1$, $y = f(-1) = \frac{-5}{3}$ a local min

∴ at $x = 3$, $y = f(3) = 9$ a local max.

e. $f''(x) = -2x + 2$

So $0 = f''(x)$

$0 = -2x + 2$

$x = 1$

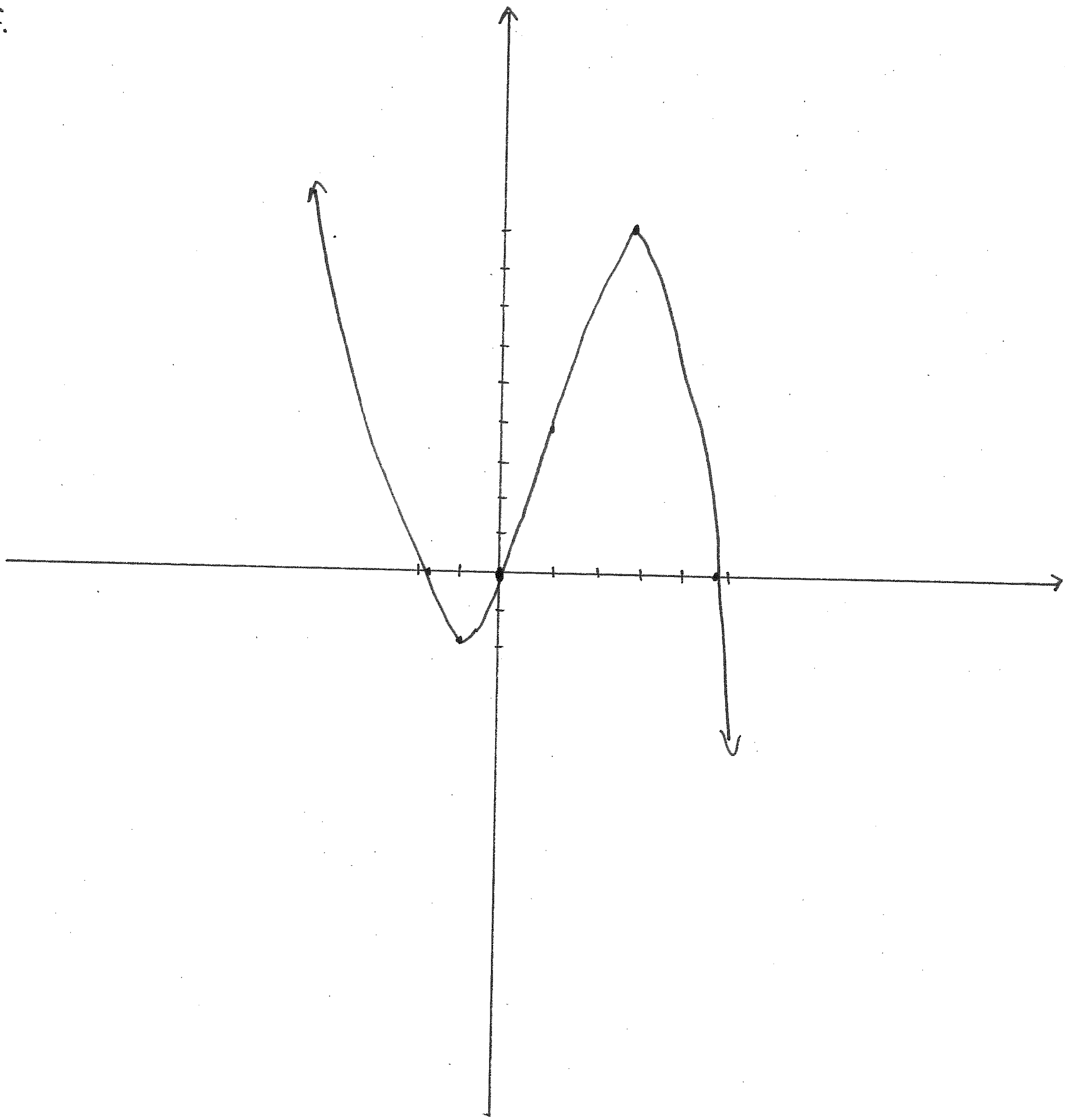
	$(-\infty, 1)$	$(1, \infty)$
p	0	2
$f''(p)$	$f''(0) = 2$ $+$	$f''(2) = -2$ $-$
concavity	$\cup$	$\cap$

$\therefore$ inflection point at

$x = 1$
$y = f(1) = \frac{11}{3}$

since concavity changes.

f.



Question 2. *(5 marks)* The quantity demanded each month of the Sicard wristwatch is related to the unit price by the equation

$$p = \frac{50}{0.01x^2+1} \quad (0 \le x \le 20)$$

where p is measured in dollars and x is measured in units of a thousand. To yield a maximum revenue, how many watches must be sold?

$R(x) = xp$

$= x\left(\frac{50}{0.01x^2+1}\right)$

$= \frac{50x}{0.01x^2+1}$

$R'(x) = \frac{50(0.01x^2+1) - 50x(0.02x)}{(0.01x^2+1)^2}$

$= \frac{-0.5x^2+50}{(0.01x^2+1)^2}$

$0 = R'(x)$

$0 = \frac{-0.5x^2+50}{(0.01x^2+1)^2}$

$50 = 0.5x^2$

$100 = x^2$

$\pm 10 = x$

$x = \cancel{-10}$ $\quad x = 10$

invalid

$R(0) = \frac{50(0)}{0.01(0)^2+1} = 0$

$R(10) = \frac{50(10)}{0.01(10)^2+1} = 250$

$R(20) = \frac{50(20)}{0.01(20)^2+1} = 200$

Question 3. *(5 marks)* Find the horizontal asymptote(s) and vertical asymptote(s) (if any) of the following function.

$$f(x) = \frac{3x^3+3x}{x^3-9x} = \frac{3x(x^2+1)}{x(x^2-9)} = \frac{3x(x^2+1)}{x(x-3)(x+3)}$$

vertical asymptote:

Possible asymptote at $x = 0, 3, -3$

$\lim_{x\to 0} f(x) = \lim_{x \to 0} \frac{3\cancel{x}(x^2+1)}{\cancel{x}(x-3)(x+3)} = \frac{3}{-9} = \frac{-1}{3}$ $\quad \therefore$ no asymptote at $x=0$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{3x(x^2+1)}{x(x-3)(x+3)} = \infty$

$\lim_{x\to 3^-}$ " " " " $= -\infty$

$\lim_{x\to -3^+}$ " " " " $= \infty$

$\lim_{x\to -3^-}$ " " " " $= -\infty$

$\therefore$ asymptote at $x = -3$
$x = 3$

horizontal asymtote:

$\lim_{x\to\pm\infty} \frac{3x^3+3x}{x^3-9x} = 3$

$\therefore$ horizontal asymptote $y = 3$

Question 4. *(5 marks)* Using logarithmic differentiation, find the derivative of the function

$$f(x) = \frac{(x^2+x+1)^3\sqrt{x+\sec x}}{\arctan x}.$$

Do not simplify (expand) your answer.

$$\text{Let} \quad y = \frac{(x^2+x+1)^3(x+\sec x)^{1/2}}{\arctan x}$$

$$\ln y = \ln\left[\frac{(x^2+x+1)^3(x+\sec x)^{1/2}}{\arctan x}\right]$$

$$\ln y = \ln(x^2+x+1)^3 + \ln(x+\sec x)^{1/2} - \ln(\arctan x)$$

$$\ln y = 3\ln(x^2+x+1) + \frac{1}{2}\ln(x+\sec x) - \ln(\arctan x)$$

$$\frac{d}{dx}\left[\text{''}\right] = \frac{d}{dx}\left[\text{''} + \text{''} - \text{''}\right]$$

$$\frac{1}{y}\cdot y' = \frac{3(2x+1)}{x^2+x+1} + \frac{1}{2}\frac{1}{x+\sec x}(1+\sec x\tan x) - \frac{1}{\arctan x}\cdot\frac{1}{(x^2+1)}$$

$$y' = y\left[\frac{3(2x+1)}{x^2+x+1} + \frac{(1+\sec x\tan x)}{2(x+\sec x)} - \frac{1}{(x^2+1)\arctan x}\right]$$

$$y' = \frac{(x^2+x+1)^3\sqrt{1+\sec x}}{\arctan x}\left[\frac{3(2x+1)}{x^2+x+1} + \frac{(1+\sec x\tan x)}{2(x+\sec x)} - \frac{1}{(x^2+1)\arctan x}\right]$$

Question 5. *(4 marks)* Find the differential of the function:

$f(x) = \sqrt{x^3 + \sqrt{x}}$

$$f'(x) = \frac{1}{2\sqrt{x^3+\sqrt{x}}}\left(3x^2 + \frac{1}{2\sqrt{x}}\right)$$

$$= \frac{\left(3x^2 + \frac{1}{2\sqrt{x}}\right)}{2\sqrt{x^3+\sqrt{x}}}$$

$$dy = f'(x)\,dx$$

$$dy = \frac{\left(3x^2 + \frac{1}{2\sqrt{x}}\right)}{2\sqrt{x^3+\sqrt{x}}}\,dx$$

Question 6. *(5 marks)* Find the derivative of the following function:

$g(x) = e^{\arcsin(x^2+\ln x)}$

$$g'(x) = e^{\arcsin(x^2+\ln x)} \cdot \frac{1}{\sqrt{1-(x^2+\ln x)^2}} \cdot \left(2x + \frac{1}{x}\right)$$

Question 7. *(2 marks)* Solve for x.

$e^{0.4t} = 8$

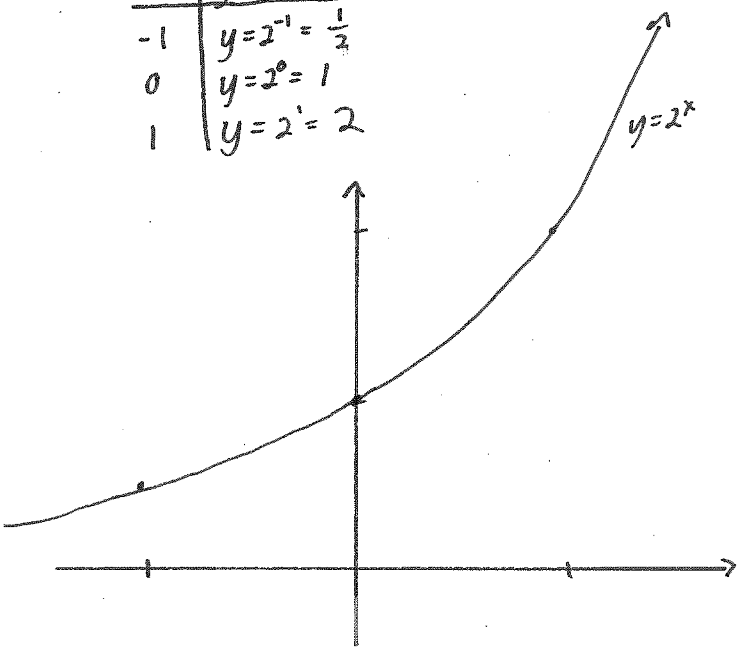
$$\ln e^{0.4t} = \ln 8$$

$$0.4t = \ln 8$$

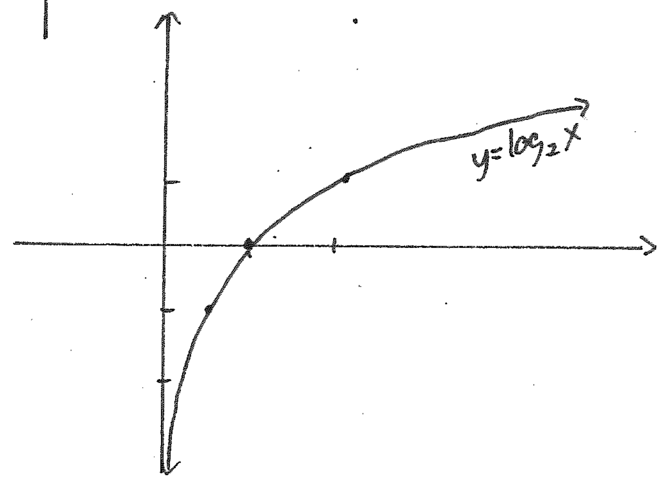
$$t = \frac{\ln 8}{0.4}$$

Question 8. *(4 marks)* Sketch $y = 2^x$ and $y = \log_2 x$.

x	$y=2^x$
-1	$y=2^{-1}=\frac{1}{2}$
0	$y=2^0=1$
1	$y=2^1=2$



x	$y=\log_2 x$
$\frac{1}{2}$	$y=\log_2(\frac{1}{2})=-1$
1	$y=\log_2(1)=0$
2	$y=\log_2(2)=1$



Question 9. *(5 marks)* Find the relative extrema, if any. Use the second derivative test, if applicable.

$f(t) = t^2 - \frac{16}{t}$ Domain $f(t)$: $\mathbb{R}\setminus\{0\}$

Lets find the critical points

$f'(t) = 2t + \frac{16}{t^2}$

$\therefore$ critical point at $t=0$ but not relative extrema since not part of domain

$0 = f'(t)$

$0 = 2t + \frac{16}{t^2}$

$-2t = \frac{16}{t^2}$

$-2t^3 = 16$

$t^3 = -8$

$t = \sqrt[3]{-8}$

$t = -2$

$f''(t) = 2 - \frac{32}{t^3}$

$f''(-2) = 2 - \frac{32}{(-2)^3}$

$= 2+4 = 6 > 0$

$\therefore$ $t=-2$ is a rel. min.

Question 10. *(5 marks)* Postal regulations specify that a parcel sent by priority mail may have a combined length and circumference of no more than 108 in. Find the dimensions of the cylindrical package of greatest volume that may be sent via priority mail. What is the volume of such a package?

$V = \pi r^2 h$ ①

$CL = h + 2\pi r$

$108 = h + 2\pi r$

$h = 108 - 2\pi r$ ②

sub ② into ①

$V = \pi r^2 (108 - 2\pi r)$

$V = 108\pi r^2 - 2\pi^2 r^3$

Lets find the critical point

$V' = 216\pi r - 6\pi^2 r^2$

So

$0 = 216\pi r - 6\pi^2 r^2$

$0 = r(216\pi - 6\pi^2 r)$

$r \neq 0$ not valid

$r = \frac{216\pi}{6\pi^2} = \frac{36}{\pi}$

Lets verify that it is a max

$V'' = 216\pi - 12\pi^2 r$

$V''\left(\frac{36}{\pi}\right) = 216\pi - 12\pi^2 \frac{36}{\pi}$

$\doteq -678.58 < 0$

$\therefore r = \frac{36}{\pi}$ is a max

$h = 108 - 2\pi\left(\frac{36}{\pi}\right) = 108 - 72$

$= 36$

$V = \pi\left(\frac{36}{\pi}\right)36$

$= 1296 \text{ in}^3$

Question 11. *(5 marks)* Suppose the number of people x going to watch Land and Freedom in a movie theater each week is related of the price p (in dollars) of the ticket by the relation

$$625p^2 + x^2 = 160\,000$$

If on a given week the price of a ticket is of \$13 and decreasing at a rate of \$0.9 per week, at what rate is the number of people going to see the movie increasing?

at a price of \$13

$625(13)^2 + x^2 = 160\,000$

$x^2 = 160\,000 - 625(13)^2$

$x = \sqrt{160\,000 - 625(13)^2}$

$x = 233.18$

Rate of change

$\frac{d}{dt}\left[625p^2 + x^2\right] = \frac{d}{dt}\left[1\,105\,625\right]$

$0 = 1250p\frac{dp}{dt} + 2x\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{-625p\frac{dp}{dt}}{x}$

$\frac{dx}{dt} = \frac{-625(13)(-0.9)}{233.18}$

$= 31.4$

$\therefore$ the number of people going to the movie is increasing at a rate of 31.4 per week.

Bonus Question. *(5 marks)* Find the derivative of the following function but do *not* simplify.

$$f(x) = x^{\sin x^{\arcsin x}}$$

Let

$$y = x^{\sin x^{\arcsin x}}$$

$$\ln y = \ln\left[x^{\sin x^{\arcsin x}}\right]$$

$$\ln y = \sin x^{\arcsin x} \ln x$$

$$\ln \ln y = \ln\left[\sin x^{\arcsin x} \ln x\right]$$

$$\ln \ln y = \ln\left[\sin x^{\arcsin x}\right] + \ln \ln x$$

$$\ln \ln y = \arcsin x \ln \sin x + \ln \ln x$$

$$\frac{d}{dx}\left[\ \text{"}\ \right] = \frac{d}{dx}\left[\ \text{"} + \text{"}\ \right]$$

$$\frac{1}{\ln y} \cdot \frac{1}{y} \cdot y' = \frac{1}{\sqrt{1-x^2}} \ln \sin x + \arcsin x \frac{1}{\sin x} \cos x + \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$y' = y \ln y \left[\frac{\ln \sin x}{\sqrt{1-x^2}} + \cot x \arcsin x + \frac{1}{x \ln x}\right]$$

$$y' = x^{\sin x^{\arcsin x}} \ln x^{\sin x^{\arcsin x}} \left[\frac{\ln \sin x}{\sqrt{1-x^2}} + \cot x \arcsin x + \frac{1}{x \ln x}\right]$$