Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S2: Winter 2012

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Test 1

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Find the average of the function $f(x) = -6x^2 + 2x + 1$ on $[-1, 2]$ using the definition of the definite integral.

$$f_{avg} \text{ on } [-1,2] = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad \Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$= \frac{1}{2-(-1)}\int_{-1}^{2} f(x)\,dx \qquad x_i = a + i\Delta x = -1 + \frac{3i}{n}$$

$$= \frac{1}{3}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \frac{1}{3}\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[-6\left(-1+\frac{3i}{n}\right)^2 + 2\left(-1+\frac{3i}{n}\right) + 1\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[-6 + 36\frac{i}{n} - \frac{54i^2}{n^2} - 2 + \frac{6i}{n} + 1\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[-7 + 42\frac{i}{n} - \frac{54i^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n} -7 + \frac{42}{n}\sum_{i=1}^{n} i - \frac{54}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[-7n + \frac{42}{\cancel{n}}\frac{\cancel{n}(n+1)}{2} - \frac{54}{n^2}\frac{\cancel{n}(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[-7 + 21\frac{(n+1)}{n} - \frac{54}{6}\frac{(n+1)}{n}\cdot\frac{(2n+1)}{n}\right]$$

$$= -7 + 21 - 9\cdot 1\cdot 2$$

$$= -7 + 21 - 18$$

$$= -4$$

Question 2. *(5 marks)* Evaluate the definite integral:

$$\int_{-\sqrt{3}}^{1} |\arctan x|\, dx = \int_{-\sqrt{3}}^{0} |\arctan x| dx + \int_{0}^{1} |\arctan x|\, dx$$

$$= -\int_{-\sqrt{3}}^{0} \arctan x\, dx + \int_{0}^{1} \arctan x\, dx$$

and since $\int \arctan x\, dx = uv - \int v\, du$ $\qquad u = \arctan x \qquad du = \frac{1}{1+x^2} dx$

$v = x \qquad dv = dx$

$$= x \arctan x - \int \frac{x}{x^2+1} dx$$

$$= x \arctan x - \frac{1}{2} \ln(x^2+1) + C$$

$$= -\left[x \arctan x - \frac{1}{2} \ln(x^2+1) \right]_{-\sqrt{3}}^{0} + \left[x \arctan x - \frac{1}{2} \ln(x^2+1) \right]_{0}^{1}$$

$$= +\left[-\sqrt{3} \arctan(-\sqrt{3}) - \frac{1}{2} \ln((-\sqrt{3})^2+1) \right] + \left[1 \arctan 1 - \frac{1}{2} \ln(1^2+1) \right]$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{1}{2} \ln 4 + \frac{\pi}{4} - \frac{1}{2} \ln 2$$

$$= \frac{4\sqrt{3}\pi + 3\pi}{12} - \ln\sqrt{8}$$

Question 3. *(5 marks)* Evaluate the definite integral:

$$\int_{-1}^{1} \frac{x^2 - x}{2x^3 - 3x^2 - 101} + \frac{x^2 \sin x}{1+x^6}\, dx$$

$$= \int_{-1}^{1} \frac{x^2 - x}{2x^3 - 3x^2 - 101} dx + \underbrace{\int_{-1}^{1} \frac{x^2 \sin x}{1+x^6} dx}_{=0}$$

since $f(x) = \frac{x^2 \sin x}{1+x^6}$ is an odd function $f(-x) = \frac{(-x)^2 \sin(-x)}{1+(-x)^6}$

$$= \frac{-x^2 \sin x}{1+x^6} = -f(x)$$

$u = 2x^3 - 3x^2 - 101$

$du = (6x^2 - 6x) dx$

$\frac{du}{6} = (x^2 - x) dx$

$u(1) = 2 \cdot 1^3 - 3 \cdot 1 - 101 = -102$

$u(-1) = 2(-1)^3 - 3(-1)^2 - 101 = -106$

$$= \int_{-106}^{-102} \frac{1}{u} \frac{du}{6} = \frac{1}{6} \Big[\ln|u| \Big]_{-106}^{-102}$$

$$= \frac{1}{6} \Big[\ln|-102| - \ln|-106| \Big]$$

$$= \ln \sqrt[6]{\frac{102}{106}} = \ln \sqrt[6]{\frac{51}{53}}$$

Question 4. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{e^x \arcsin(e^x - 1)}{\sqrt{e^{2x} - 2e^x}}\,dx \quad = \int u\,du \quad = \frac{u^2}{2} + C$$

$$= \frac{\left(\arcsin(e^x-1)\right)^2}{2} + C$$

$$u = \arcsin(e^x - 1)$$

$$du = \frac{1 \cdot e^x}{\sqrt{1-(e^x-1)^2}}\,dx$$

$$du = \frac{1 \cdot e^x}{\sqrt{1-[e^{2x}-2e^{2x}+1]}}\,dx$$

$$du = \frac{e^x}{\sqrt{2e^x - e^{2x}}}\,dx$$

Question 5. *(5 marks)* Evaluate the expression and simplify:

$$\frac{d}{dx}\left[\underbrace{\int_{\ln(2x)}^{\ln(x^2)} u^{e^u}\,du}_{h(x)}\right]$$

$$h(x) = \int_{\ln 2x}^{\ln x^2} u^{e^u}\,du = \int_{\ln 2x}^{0} u^{e^u}\,du + \int_{0}^{\ln x^2} u^{e^u}\,du$$

$$= -\int_{0}^{\ln 2x} u^{e^u}\,du + \int_{0}^{\ln x^2} u^{e^u}\,du$$

$$= -f(g_1(x)) + f(g_2(x))$$

where $f(x) = \int_0^x u^{e^u}\,du$

$g_1(x) = \ln 2x$

$g_2(x) = \ln x^2$

$$\therefore \quad h'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x)$$

$f'(x) = x^{e^x}$ by 2^{nd} FTC

$g_1'(x) = \frac{1}{x}$

$g_2'(x) = \frac{2}{x}$

$$= -(\ln 2x)^{e^{\ln 2x}} \cdot \frac{1}{x} + (\ln x^2)^{e^{\ln x^2}} \cdot \frac{2}{x}$$

$$= \frac{-(\ln 2x)^{2x}}{x} + \frac{2(\ln x^2)^{x^2}}{x}$$

Question 6. *(5 marks)* Evaluate the indefinite integral:

$$\int x \csc^2 2x \, dx = uv - \int v \, du$$

$u = x \qquad du = dx$

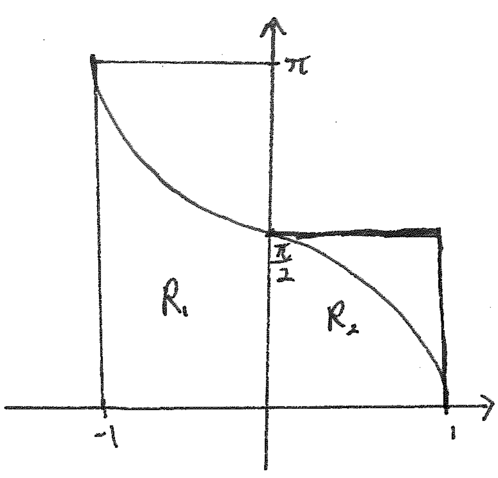
$v = \frac{-\cot 2x}{2} \qquad dv = \csc^2 2x \, dx$

$$= \frac{-x \cot 2x}{2} - \int \frac{-\cot 2x}{2} \cdot \cdot \; dx$$

$$= \frac{-x \cot 2x}{2} + \frac{1}{2}\int \cot 2x \, dx$$

$$= \frac{-x \cot 2x}{2} + \frac{1}{4} \ln|\sin 2x| + C$$

Question 7. *(5 marks)* Estimate the area under the graph of $f(x) = \arccos x$ from $x = -1$ to $x = 1$ using two rectangles and using the left end points. Sketch the curve and the approximating rectangles. Is the estimate an overestimate or an underestimate? Justify.



$$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{2} = 1$$

$$\text{Area} \approx R_1 + R_2$$

$$= \pi \cdot 1 + \frac{\pi}{2} \cdot 1$$

$$= \frac{3\pi}{2}$$

The estimate is an overestimate since $f(x)$ is deacreasing

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} < 0 \text{ on } (-1, 1).$$

Question 8. *(5 marks)* Prove: If $f(x)$ is an even integrable function on $[-a,a]$ then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

since $f(x)$ is even $f(-x)=f(x)$

$$\begin{aligned}
LHS &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \\
&= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)dx \\
&= -\int_{0}^{-a} f(-x)\,dx + \int_{0}^{a} f(x)dx \\
&= -\int_{0}^{a} f(u)\,(-du) + \int_{0}^{a} f(x)dx \\
&= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)dx \\
&= 2\int_{0}^{a} f(u)\,du.
\end{aligned}$$

$u=-x$
$du=-dx$
$-du=dx$

$u(-a)=-(-a)=a$
$u(0)=-0=0$

Question 9.

a. *(1 mark)* Show that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

b. *(4 marks)* If f and g are inverse functions *(that is, $f(g(x))=x$ and $g(f(x))=x$)* and f' is continuous, prove that

$$\int_{a}^{b} f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

(hint: you may use part a.)

a) $\int f(x)dx = uv - \int v\,du$
$\quad = xf(x) - \int xf'(x)dx$

$u=f(x)$ $\quad du=f'(x)dx$
$v=x$ $\quad dv=dx$

b) using part a.

$$\int_{a}^{b} f(x)dx = \Big[xf(x)\Big]_{a}^{b} - \int_{a}^{b} xf'(x)dx$$

$$= bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)dy$$

$y=f(x)$ $\quad \Leftrightarrow \quad g(y)=g(f(x))$
$dy=f'(x)dx$ $\quad g(y)=x$
$y(b)=f(b)$
$y(a)=f(a)$

Bonus Question. *(3 marks)*
Evaluate:

$$\lim_{h\to 0}\frac{\lim_{n\to\infty}\left[\frac{x+h-\pi}{n}\left[\left(\pi+\frac{x+h-\pi}{n}\right)^e+\left(\pi+2\frac{x+h-\pi}{n}\right)^e\ldots+\left(\pi+n\frac{x+h-\pi}{n}\right)^e\right]\right]-\lim_{n\to\infty}\left[\frac{x-\pi}{n}\left[\left(\pi+\frac{x-\pi}{n}\right)^e+\left(\pi+2\frac{x-\pi}{n}\right)^e+\ldots+\left(\pi+n\frac{x-\pi}{n}\right)^e\right]\right]}{h}$$

$$= \lim_{h\to 0}\frac{\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\pi+i\,\frac{x+h-\pi}{n}\right)^e\left[\frac{x+h-\pi}{n}\right]\right]-\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\pi+i\,\frac{x-\pi}{n}\right)^e\left[\frac{x-\pi}{n}\right]\right]}{h}$$

$$= \lim_{h\to 0}\frac{\int_{\pi}^{x+h} t^e\,dt - \int_{\pi}^{x} t^e\,dt}{h}$$

$$= \frac{d}{dx}\left[\int_{\pi}^{x} t^e\,dt\right]$$

$= x^e$ by 2^{nd} FTC