Dawson College: Calculus II (SCIENCE): 201-NYB-05-S2: Winter 2012

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Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Evaluate the integral:

$$\int e^{\pi x}\tan^3(e^{\pi x})\sec^5(e^{\pi x})\,dx = \int \tan^3 u \sec^5 u \,\frac{du}{\pi}$$

$u = e^{\pi x}$

$du = \pi e^{\pi x}\,dx$

$\frac{du}{\pi} = e^{\pi x}\,dx$

$$= \frac{1}{\pi}\int \tan^2 u \sec^4 u \tan u \sec u \, du$$

$$= \frac{1}{\pi}\int (\sec^2 u - 1)\sec^4 u \tan u \sec u \, du$$

$v = \sec u$

$dv = \sec u \tan u \, du$

$$= \frac{1}{\pi}\int (v^2-1)v^4\,dv$$

$$= \frac{1}{\pi}\int v^6 - v^4\,dv$$

$$= \frac{1}{\pi}\left[\frac{v^7}{7} - \frac{v^5}{5}\right] + C$$

$$= \frac{1}{\pi}\left[\frac{\sec^7 u}{7} - \frac{\sec^5 u}{5}\right] + C$$

$$= \frac{1}{\pi}\left[\frac{\sec^7 e^{\pi x}}{7} - \frac{\sec^5 e^{\pi x}}{5}\right] + C$$

Question 2. *(5 marks)* Evaluate the integral:

$$\int \frac{x^2+2bx+b^2}{(a^2-b^2-2bx-x^2)^{3/2}}\,dx = \int \frac{(x+b)^2}{\left(a^2-[b^2+2bx+x^2]\right)^{3/2}}\,dx$$

$$= \int \frac{(x+b)^2}{\left(a^2-(x+b)^2\right)^{3/2}}\,dx$$

$u = x+b$

$du = dx$

$$= \int \frac{u^2}{(a^2-u^2)^{3/2}}\,du$$

$u = a\sin\theta$

$du = a\cos\theta\,d\theta$

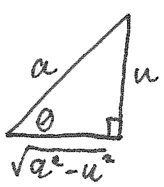
$$= \int \frac{(a\sin\theta)^2}{\left(a^2-(a\sin\theta)^2\right)^{3/2}}\,a\cos\theta\,d\theta$$

$\frac{u}{a} = \sin\theta$

$\theta = \arcsin\frac{u}{a}$

$$= a^3\int \frac{\sin^2\theta\cos\theta}{\left(a^2(1-\sin^2\theta)\right)^{3/2}}\,d\theta$$

$$= a^3\int \frac{\sin^2\theta\cos\theta}{\left(a^2\cos^2\theta\right)^{3/2}}\,d\theta$$



$$= \frac{a^3}{a^3}\int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta}$$

$\therefore \tan\theta = \frac{u}{\sqrt{a^2-u^2}}$

$$= \int \tan^2\theta\,d\theta$$

$$= \int \sec^2\theta - 1\,d\theta$$

$$= \tan\theta - \theta + C$$

$$= \frac{u}{\sqrt{a^2-u^2}} - \arcsin\frac{u}{a} + C$$

$$= \frac{x+b}{\sqrt{a^2-(x+b)^2}} - \arcsin\left(\frac{x+b}{a}\right) + C$$

Question 3. *(5 marks)* Evaluate the integral:

$$\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx$$

$$\frac{x^3-2x^2+x+1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$x^3-2x^2+x+1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

Let $x=i$: $\quad i^3-2i^2+i+1 = (Ai+B)(i^2+4) + (Ci+D)\underbrace{(i^2+1)}_{0}$

$$-i-2(-1)+i+1 = (Ai+B)(3)$$

$$3 = 3B+3Ai$$

$$\therefore B=1 \text{ and } A=0$$

Let $x=2i$: $\quad (2i)^3-2(2i)^2+2i+1 = (A(2i)+B)\underbrace{((2i)^2+4)}_{0} + (C(2i)+D)((2i)^2+1)$

$$8i^3-2(-4)+2i+1 = (2Ci+D)(-3)$$

$$-8i+8+2i+1 = -6Ci-3D$$

$$9-6i = -3D-6Ci$$

$$\therefore D=-3 \text{ and } C=1$$

$$\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx = \int \frac{1}{x^2+1}\,dx + \int \frac{x}{x^2+4}\,dx + \int \frac{-3}{x^2+4}\,dx$$

$$= \arctan x + \frac{1}{2}\ln(x^2+4) - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + C$$

Question 4. *(5 marks)* Evaluate the integral:

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2\phi}{\sqrt{1-\tan^2\phi}}\,d\phi$$

infinite discontinuity at $x=\frac{\pi}{4}$

$$= \lim_{b\to\frac{\pi}{4}^-} \int_0^b \frac{\sec^2\phi}{\sqrt{1-\tan^2\phi}}\,d\phi$$

$u=\tan\phi$
$du=\sec^2\phi\,d\phi$
$u(b)=\tan b$
$u(0)=\tan 0=0$

$$= \lim_{b\to\frac{\pi}{4}^-} \int_0^{\tan b} \frac{du}{\sqrt{1-u^2}}$$

$$= \lim_{b\to\frac{\pi}{4}^-} \Big[\arcsin u\Big]_0^{\tan b}$$

$$= \lim_{b\to\frac{\pi}{4}^-} \Big[\arcsin \tan b - \arcsin 0\Big]$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Question 5. *(5 marks)* Find the values of κ for which the integral converges and evaluate the integral for those values of κ.

$$\int_1^\infty x^\kappa \ln x\, dx = \lim_{b\to\infty} \int_1^b x^\kappa \ln x\, dx \qquad u = \ln x \quad du = \frac{1}{x}dx \qquad v = \frac{x^{\kappa+1}}{\kappa+1} \quad dv = x^\kappa dx$$

if $\kappa \neq -1$

$$= \lim_{b\to\infty} \left[[uv]_1^b - \int_1^b v\,du \right]$$

$$= \lim_{b\to\infty} \left[\left[\frac{x^{\kappa+1}\ln x}{\kappa+1}\right]_1^b - \int_1^b \frac{x^{\kappa+1}}{\kappa+1}\frac{1}{x}dx \right]$$

$$= \lim_{b\to\infty} \left[\frac{b^{\kappa+1}\ln b}{\kappa+1} - \frac{1^{\kappa+1}\ln 1}{\kappa+1} - \int_1^b \frac{x^\kappa}{\kappa+1}dx \right]$$

$$= \lim_{b\to\infty} \left[\frac{b^{\kappa+1}\ln b}{\kappa+1} - \left[\frac{x^{\kappa+1}}{(\kappa+1)^2}\right]_1^b \right]$$

$$= \lim_{b\to\infty} \left[\frac{b^{\kappa+1}\ln b}{\kappa+1} - \left[\frac{b^{\kappa+1}}{(\kappa+1)^2} - \frac{1^{\kappa+1}}{(\kappa+1)^2}\right] \right]$$

$$= \lim_{b\to\infty} \left[\frac{b^{\kappa+1}\ln b}{\kappa+1} - \frac{b^{\kappa+1}}{(\kappa+1)^2} + \frac{1}{(\kappa+1)^2} \right]$$

$$= \lim_{b\to\infty} \left[\frac{b^{\kappa+1}}{\kappa+1}\left[\ln b - \frac{1}{\kappa+1}\right] \right] + \frac{1}{(\kappa+1)^2}$$

$$= \lim_{b\to\infty} \left[\frac{b^{\kappa+1}}{\kappa+1}\left[\ln b - \frac{\ln e}{\kappa+1}\right] \right] + \frac{1}{(\kappa+1)^2}$$

$$= \lim_{b\to\infty} \left[\frac{b^{\kappa+1}}{\kappa+1}\left[\ln b - \ln e^{\frac{1}{\kappa+1}}\right] \right] + \frac{1}{(\kappa+1)^2}$$

$$= \lim_{b\to\infty} \left[\frac{b^{\kappa+1}}{\kappa+1}\left[\ln \frac{b}{e^{\frac{1}{\kappa+1}}}\right] \right] + \frac{1}{(\kappa+1)^2}$$

if $\kappa > -1$ then $\lim_{b\to\infty} \left[\frac{b^{\kappa+1}}{\kappa+1} \ln\left(\frac{b}{e^{\frac{1}{\kappa+1}}}\right) \right]$ diverges to ∞

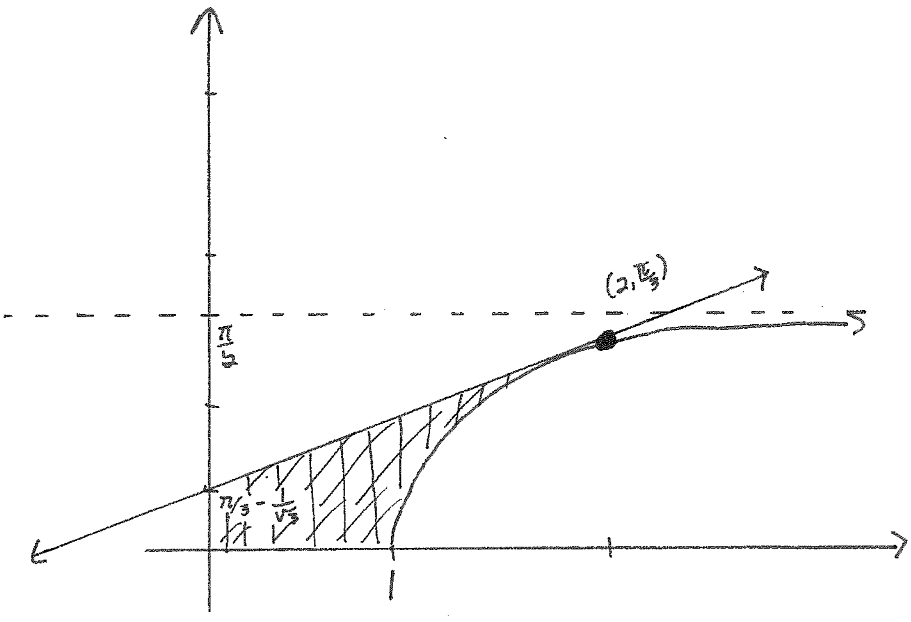
if $\kappa < -1$ then $\lim_{b\to\infty} \left[\frac{b^{\kappa+1}}{\kappa+1} \ln\left(\frac{b}{e^{\frac{1}{\kappa+1}}}\right) \right]$ I.f. $0\cdot\infty$

$$= \frac{1}{\kappa+1}\lim_{b\to\infty} \left[\frac{\ln\left(\frac{b}{e^{1/(\kappa+1)}}\right)}{b^{-\kappa-1}} \right] \quad \text{I.f. } \frac{\infty}{\infty}$$

$$\overset{\hat{H}}{=} \frac{1}{\kappa+1}\lim_{b\to\infty} \left[\frac{\frac{1}{b}}{(-\kappa-1)b^{-\kappa-2}} \right] = 0$$

$\therefore$ integral converges to $\frac{1}{(\kappa+1)^2}$.

Question 6. *(5 marks)* Sketch the region in the first quadrant enclosed by $f(x) = \operatorname{arcsec} x$, the tangent line to $f(x)$ at $(2, \frac{\pi}{3})$, the x-axis and y-axis. Set up the integral to find the area enclosed but *do not* evaluate.



$f'(x) = \frac{1}{x\sqrt{x^2-1}}$

slope of tangent at $x=2$ $= f'(2)$

$= \frac{1}{2\sqrt{2^2-1}}$

$= \frac{1}{2\sqrt{3}}$

So $y = mx + b$

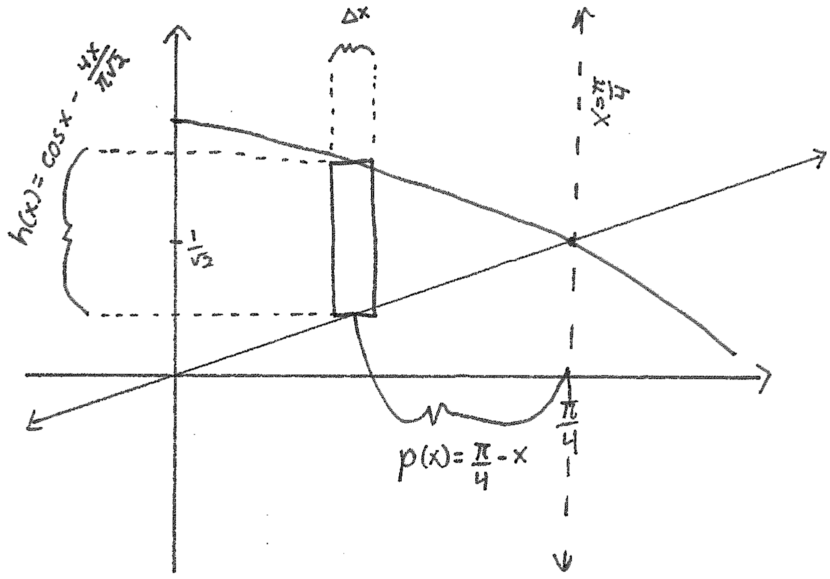
$\frac{\pi}{3} = \frac{1}{2\sqrt{3}}(2) + b$

$b = \frac{\pi}{3} - \frac{1}{\sqrt{3}}$

$\therefore y = \frac{1}{2\sqrt{3}}x + \frac{\pi}{3} - \frac{1}{\sqrt{3}}$

Area $= \int_0^1 \frac{x}{2\sqrt{3}} + \frac{\pi}{3} - \frac{1}{\sqrt{3}}\, dx + \int_1^2 \frac{x}{2\sqrt{3}} + \frac{\pi}{3} - \frac{1}{\sqrt{3}} - \operatorname{arcsec} x\, dx$

Question 7. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region in the first quadrant bounded by the graphs of $y = \cos x$, $y = \frac{4x}{\pi\sqrt{2}}$ and $x = 0$ rotated about the line $x = \frac{\pi}{4}$.



representative element:

$$\Delta V = 2\pi p(x) h(x) \Delta x$$

$$= 2\pi \left(\frac{\pi}{4} - x\right)\left(\cos x - \frac{4x}{\pi\sqrt{2}}\right)\Delta x$$

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \left(\frac{\pi}{4} - x_i\right)\left(\cos x_i - \frac{4x_i}{\pi\sqrt{2}}\right)\Delta x_i = \int_0^{\pi/4} 2\pi \left(\frac{\pi}{4} - x\right)\left(\cos x - \frac{4x}{\pi\sqrt{2}}\right)dx$$

Question 8. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $x = y^2 - 2y$, $x = y$ rotated about the line $x = -1$.

Intersection of two curve:

$$y = y^2 - 2y$$
$$0 = y^2 - 3y$$
$$0 = y(y-3)$$
$$y=0 \quad y=3$$

vertex of: $x = y^2 - 2y$

$$x = y^2 - 2y + 1 - 1$$
$$x = (y-1)^2 - 1$$

$\therefore$ vertex at $(-1, 1)$

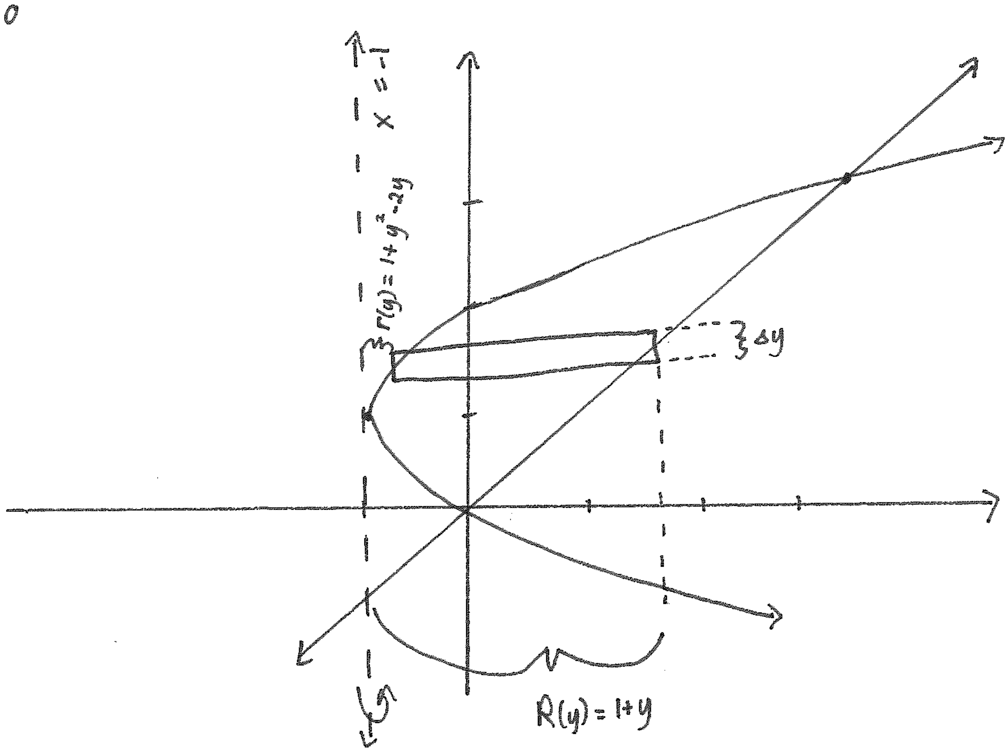
representative element:

$$\Delta V = \pi \left[(R(y))^2 - (r(y))^2 \right] \Delta y$$
$$= \pi \left[(1+y)^2 - (y^2 - 2y + 1) \right] \Delta y$$

y-int: $0 = y^2 - 2y$

$$0 = y(y-2)$$
$$y=0 \quad y=2$$

x-int: $x = 0$



$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi \left[(1+y_i)^2 - (y_i^2 - 2y_i + 1)^2 \right] \Delta y_i$$

$$= \int_0^3 \pi \left[(1+y)^2 - (y^2 - 2y + 1)^2 \right] dy$$

Question 9. *(5 marks)* Find the length of the curve $y = \ln(\sec x)$ on $[0, \frac{\pi}{4}]$.

$$s = \int_0^{\pi/4} \sqrt{1 + (y')^2}\, d$$

$$y' = \frac{1}{\sec x} \sec x \tan x = \tan x$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/4} |\sec x|\, dx$$

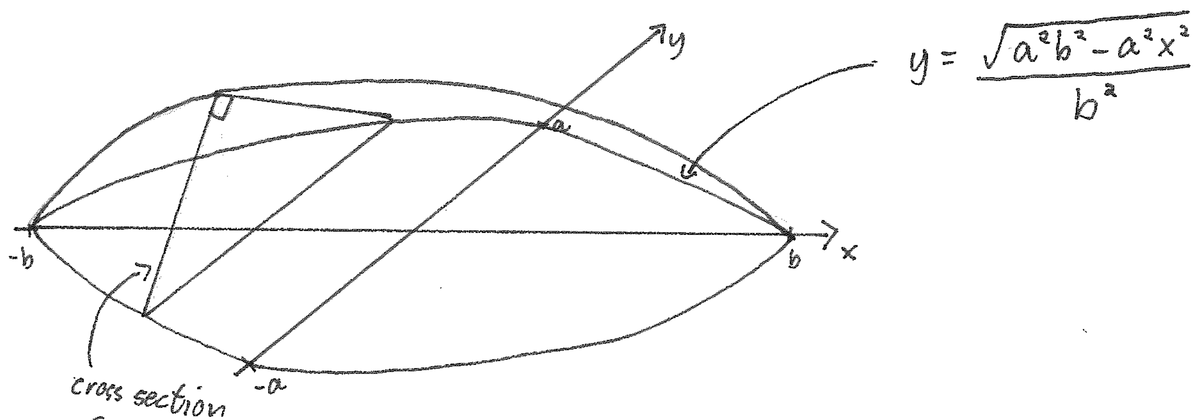
$$= \int_0^{\pi/4} \sec x\, dx$$

$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

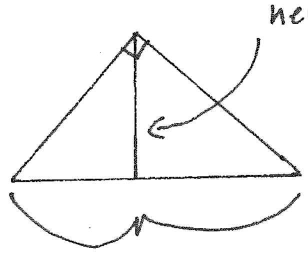
$$= \ln|\sec \tfrac{\pi}{4} + \tan \tfrac{\pi}{4}| - \ln|\sec 0 + \tan 0|$$

$$= \ln(\sqrt{2} + 1)$$

Bonus Question. *(5 marks)* Find the volume of a solid whose base is an elliptical region with boundary curve $a^2x^2 + b^2y^2 = a^2b^2$. Cross-sections perpendicular to the x-axis are isoceles right triangles with hypotenuse in the base.



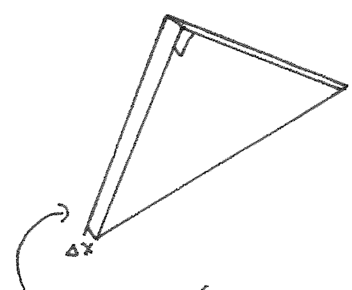
$$\text{height} = \tfrac{1}{2}\,\text{base} = \frac{\sqrt{a^2b^2 - a^2b^2}}{b^2}$$



$$\text{base} = \frac{2\sqrt{a^2b^2 - a^2x^2}}{b^2}$$

Area of cross section

$$= \tfrac{1}{2}(\text{base})(\text{height}) = \frac{a^2b^2 - a^2x^2}{b^4}$$



Volume = (Area of cross section)Δx

$$= \left(\frac{a^2b^2 - a^2x^2}{b^4}\right)\Delta x$$

$$\text{Volume} = \lim_{\max|\Delta x_i| \to 0} \sum_{i=1}^{n} \left(\frac{a^2b^2 - a^2x_i^2}{b^4}\right)\Delta x_i$$

$$= \int_{-b}^{b} \frac{a^2b^2 - a^2x^2}{b^4}\,dx$$

$$= 2\int_{0}^{b} \frac{a^2b^2 - a^2x^2}{b^4}\,dx = \frac{2}{b^4}\left[a^2b^2x - \frac{a^2x^3}{3}\right]_0^b$$

$$= \frac{2}{b^4}\left[a^2b^2b - \frac{a^2b^3}{3}\right]$$

$$= \frac{4a^2b^3}{3b^4} = \frac{4}{3}\frac{a^2}{b}$$