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Test 3

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.

a. *(4 marks)* Show that the series

$$\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$$

is convergent

b. *(1 mark)* Deduce that

$$\lim_{n\to\infty} \frac{n^n}{(2n)!} = 0.$$

Lets apply the ratio test. Let $a_n = \dfrac{n^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}}{(2(n+1))!}}{\frac{n^n}{(2n)!}}\right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)^n \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\, n^n}$$

$$= \frac{1}{2}\lim_{n\to\infty} \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}(2n+1)\, n^n}$$

$$= \frac{1}{2}\lim_{n\to\infty}\frac{1}{2n+1}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \frac{1}{2}\lim_{n\to\infty}\frac{1}{2n+1}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$= \frac{1}{2}\lim_{n\to\infty}\frac{1}{2n+1}\cdot e$$

$$= 0 < 1$$

$\therefore$ converge by ratio test

Let $f(x) = \left(1+\frac{1}{x}\right)^x$

$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ I.F. 1^∞

$\ln y = \ln \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$\ln y = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x$

$\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$ I.F. $\infty\cdot 0$

$\ln y = \lim_{x\to\infty} \dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$\ln y = \lim_{x\to\infty} \dfrac{\frac{1}{1+\frac{1}{x}}\cdot \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}}$

$\ln y = 1$

$y = e$

b) Since $\sum a_n$ converges

then $\lim\limits_{n\to\infty} a_n = 0.$

$$\lim_{n\to\infty}\frac{n^n}{(2n)!} = 0$$

Question 2. *(5 marks)* Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1921}^{\infty} \frac{2^n(x-7)^n}{\sqrt{n-17}}$$

Let $a_n(x) = \dfrac{2^n(x-7)^n}{\sqrt{n-17}}$

Lets determine the radius of convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}(x-7)^{n+1}}{\sqrt{n+1-17}}}{\frac{2^n(x-7)^n}{\sqrt{n-17}}}\right| = \lim_{n\to\infty}\left|2(x-7)\cdot\sqrt{\frac{n-17}{n-16}}\right|$$

$= 2|x-7| < 1$ converges if

So

$2|x-7| < 1$

$|x-7| < \frac{1}{2} = R$

$\therefore$ radius of convergence $R = \frac{1}{2}$

$-\frac{1}{2} < x-7 < \frac{1}{2}$

$7-\frac{1}{2} < x < \frac{1}{2}+7 = \frac{15}{2}$

$7-\frac{1}{2} = \frac{13}{2}$

Lets test endpoint for interval of convergence

Let $x = \frac{13}{2}$

$$\sum_{n=1921}^{\infty}\frac{2^n\left(\frac{13}{2}-7\right)^n}{\sqrt{n-17}} = \sum_{n=1921}^{\infty}\frac{2^n\left(\frac{-1}{2}\right)^n}{\sqrt{n-17}} = \sum_{n=1921}^{\infty}\frac{(-1)^n}{\sqrt{n-17}} = \sum_{n=1921}^{\infty}(-1)^n b_n$$

where $b_n = \dfrac{1}{\sqrt{n-17}}$

(i) $b_{n+1} \overset{?}{\leq} b_n$

$$\frac{1}{\sqrt{n+1-17}} \leq \frac{1}{\sqrt{n-17}}$$

$\sqrt{n-17} \leq \sqrt{n-16}$

$n-17 \leq n-16$

$-17 \leq -16$ ✓

(ii) $\lim_{n\to\infty} b_n = 0$

$\therefore$ converges by alternating series test

$\therefore$ converges at $x = \frac{13}{2}$

Let $x = 15/2$

$$\sum_{n=1921}^{\infty}\frac{2^n\left(\frac{15}{2}-7\right)^n}{\sqrt{n-17}} = \sum_{n=1921}^{\infty}\frac{2^n\left(\frac{1}{2}\right)^n}{\sqrt{n-17}} = \sum_{n=1921}^{\infty}\frac{1}{\sqrt{n-17}}$$

$$\sum_{n=1921}^{\infty}\frac{1}{\sqrt{n-17}} = \sum_{n=1921}^{\infty} a_n$$

$a_n \geq \dfrac{1}{\sqrt{n}} = b_n \geq 0$

$\therefore \sum_{n=1921}^{\infty} b_n$ diverges since p-series where $p = \frac{1}{2} < 1$

$\therefore \sum_{n=1921}^{\infty} a_n$ diverges by comparison test.

$\therefore$ does not converge at $x = 15/2$

$\therefore$ interval of convergence

$\left[\frac{13}{2}, \frac{15}{2}\right)$

Question 3. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=3}^{\infty} \ln\left(\frac{\sec\left(\frac{\pi}{n}\right)}{\sec\left(\frac{\pi}{n+1}\right)}\right) = \sum_{n=3}^{\infty}\left[\ln \sec\left(\frac{\pi}{n}\right) - \ln \sec\left(\frac{\pi}{n+1}\right)\right]$$

Lets look at the partial sum

$$S_n = a_3 + a_4 + a_5 + \dots + a_{n-2} + a_{n-1} + a_n$$

$$= \left[\ln \sec\left(\frac{\pi}{3}\right) - \cancel{\ln \sec\left(\frac{\pi}{4}\right)}\right] + \left[\cancel{\ln \sec\left(\frac{\pi}{4}\right)} - \cancel{\ln \sec\left(\frac{\pi}{5}\right)}\right]$$

$$+ \left[\cancel{\ln \sec\left(\frac{\pi}{5}\right)} - \cancel{\ln \sec\left(\frac{\pi}{6}\right)}\right] + \dots + \left[\cancel{\ln \sec\left(\frac{\pi}{n-2}\right)} - \cancel{\ln \sec\left(\frac{\pi}{n-1}\right)}\right]$$

$$+ \left[\cancel{\ln \sec\left(\frac{\pi}{n-1}\right)} - \cancel{\ln \sec\left(\frac{\pi}{n}\right)}\right] + \left[\cancel{\ln \sec\left(\frac{\pi}{n}\right)} - \ln \sec\left(\frac{\pi}{n+1}\right)\right]$$

$$= \ln \sec\left(\frac{\pi}{3}\right) - \ln \sec\left(\frac{\pi}{n+1}\right)$$

$$S = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left[\ln \sec\left(\frac{\pi}{3}\right) - \ln \sec\left(\frac{\pi}{n+1}\right)\right]$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

Question 4. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=4}^{\infty} \frac{(-1)^n \tan\left(\frac{\pi}{n}\right)}{1+(1.1)^n}$$

Absolute Convergence

$$\sum_{n=4}^{\infty} \left| \frac{(-1)^n \tan\left(\frac{\pi}{n}\right)}{1+(1.1)^n} \right| = \sum_{n=4}^{\infty} \frac{\tan\left(\frac{\pi}{n}\right)}{1+(1.1)^n} \qquad \text{Let } a_n = \frac{\tan\left(\frac{\pi}{n}\right)}{1+(1.1)^n}$$

$$0 \le a_n \le \frac{1}{1+(1.1)^n} < \frac{1}{(1.1)^n} = \left(\frac{10}{11}\right)^n = b_n$$

$\therefore \sum_{n=4}^{\infty} a_n$ is convergent by comparison test since

$\sum_{n=4}^{\infty} b_n$ is convergent (geometric series where $|r| = \frac{10}{11} < 1$)

$\therefore$ the series is absolutely convergent.

Question 5. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n!\tan\left(\frac{1}{n}\right)}{(n-1)!}$$

$$\text{Let } a_n = \frac{n!\tan\left(\frac{1}{n}\right)}{(n-1)!} = \frac{\cancel{(n-1)!}\, n\tan\left(\frac{1}{n}\right)}{\cancel{(n-1)!}} = n\tan\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) \qquad \text{Let } f(x) = x\tan\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{1/x} \qquad \text{I.f. } \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{x}\right)\cancel{\frac{-1}{x^2}}}{\cancel{-1/x^2}}$$

$$= 1 \neq 0$$

$\therefore$ diverges by n^{th} term divergence test

Question 6.

a. *(1 mark)* Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$\left\{\frac{9}{\pi}, \frac{16}{\pi^2}, \frac{25}{\pi^3}, \frac{36}{\pi^4}, \frac{49}{\pi^5}, \ldots\right\}_{n=1}^{\infty}$$

b. *(1 mark)* Show that a_n is monotonic.

c. *(1 mark)* Show that a_n is bounded.

d. *(1 mark)* By which theorem can we conclude that a_n converges.

e. *(1 mark)* Determine the limit of a_n as $n \to \infty$.

a. $a_n = \frac{(n+2)^2}{\pi^n}$

b. Lets show that a_n is decreasing. Let $f(x) = \frac{(x+2)^2}{\pi^x}$

$$f'(x) = \frac{2(x+2)\pi^x - (x+2)^2\pi^x \ln\pi}{\pi^{2x}}$$
$$= \frac{\pi^x(x+2)\left[2-(x+2)\ln\pi\right]}{\pi^{2x}} < 0 \quad \text{for } x \geq 1$$

$\therefore$ $f(x)$ is decreasing

$\therefore$ $f(n+1) < f(n)$

$a_{n+1} < a_n$

$\therefore$ a_n is monotonic decreasing.

c. Since a_n is decreasing $a_1 > a_n$

$\therefore$ $M = a_1 = \frac{9}{\pi}$ is an upper bound.

and bounded below by $0 = m$ since always positive.

d. Monotonic Sequence Theorem.

e. Let $f(x) = \frac{(x+2)^2}{\pi^x}$

$\lim_{x\to\infty} \frac{(x+2)^2}{\pi^x}$ i.f. $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2(x+2)}{\pi^x \ln\pi}$ by $\hat{H}$ i.f. $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2}{\pi^x(\ln\pi)^2}$ by $\hat{H}$

$= 0$

Question 7. *(5 marks)* Find the values of κ ≥ 0 for which the series is convergent.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\kappa}}$$

Let $f(x) = \dfrac{1}{x(\ln x)^{K}}$

- $f(x)$ is positive for $x \geq 2$
- $f(x)$ is continuous for $x \geq 2$
- Is $f(x)$ decreasing for $x \geq 2$?

$$f'(x) = \frac{0(x(\ln x)^K) - [(\ln x)^K + (K)(\ln x)^{K-1}]}{(x(\ln x)^K)^2}$$

$$= \frac{-(\ln x)^K\left[1 + \frac{K}{(\ln x)}\right]}{x^2(\ln x)^{2K}} < 0$$

$\therefore$ decreases for $x \geq 2$.

$$\int_2^{\infty} \frac{1}{x(\ln x)^K}\,dx$$

$$= \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^K}\,dx$$

$u = \ln x$

$du = \frac{1}{x}dx$

$u(b) = \ln b$

$u(2) = \ln 2$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^K}\,dx$$

if $K = 1$

$$= \lim_{b\to\infty} \Big[\ln|u|\Big]_{\ln 2}^{\ln b} = \lim_{b\to\infty} \ln\ln b - \ln\ln 2 = \infty$$

$\therefore$ diverges $\therefore$ series diverges for $K=1$ by the integral test

if $K \neq 1$

$$= \lim_{b\to\infty} \left[\frac{u^{-K+1}}{-K+1}\right]_{\ln 2}^{\ln b} = \lim_{b\to\infty} \left[\frac{(\ln b)^{-K+1}}{-K+1} - \frac{(\ln 2)^{-K+1}}{-K+1}\right]$$

If $0<K<1$ then integral diverges $\therefore$ series diverges for $0<K<1$. by integral test

If $K>1$ then integral converges $\therefore$ series converges for $K>1$ by integral test.

Question 8. *(5 marks)* What is the value of ζ if

$$\sum_{n=1}^{\infty} \sqrt{2}(\ln \zeta)^n = \pi$$

Let $a_n = \sqrt{2}\,(\ln z)^n$

$$\pi = \sum_{n=2}^{\infty} \sqrt{2}\,(\ln z)^n$$

$$\pi = \sum_{n=0}^{\infty} \sqrt{2}\,(\ln z)^n \quad - a_0$$

$$\pi = \frac{\sqrt{2}}{1-\ln z} - \sqrt{2}$$

$$\pi(1-\ln z) = \sqrt{2} - \sqrt{2}\,(1-\ln z)$$

$$\pi - \pi \ln z = \sqrt{2} - \sqrt{2} + \sqrt{2}\ln z$$

$$\pi = \sqrt{2}\ln z + \pi \ln z$$

$$\pi = (\sqrt{2}+\pi)\ln z$$

$$\ln z = \frac{\pi}{\sqrt{2}+\pi}$$

$$z = e^{\frac{\pi}{\sqrt{2}+\pi}}$$

$|r| = |\ln z| = \frac{\pi}{\sqrt{2}+\pi} < 1 \quad \therefore$ convirges to π

Question 9. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1871}^{\infty} \frac{(-1)^n(n+1)}{\sqrt{n^4-2n^2+1}} = \sum_{n=1871}^{\infty} \frac{(-1)^n(n+1)}{\sqrt{(n^2-1)^2}} = \sum_{n=1871}^{\infty} \frac{(-1)^n(n+1)}{n^2-1} = \sum_{n=1871}^{\infty} \frac{(-1)^n \cancel{(n+1)}}{\cancel{(n+1)}(n-1)}$$

$$= \sum_{n=1871}^{\infty} (-1)^n \frac{1}{(n-1)}$$

<u>absolute convergence:</u>

$$\sum_{n=1871}^{\infty} \left| \frac{(-1)^n}{(n-1)} \right| = \sum_{n=1871}^{\infty} \frac{1}{n-1} = \sum_{n=1871}^{\infty} a_n$$

$a_n \geq \frac{1}{n} = b_n \geq 0$ $\quad \sum_{n=1871}^{\infty} a_n$ diverges by comparison test since $\sum_{n=1871}^{\infty} b_n$ diverges (p-series where $p = \frac{1}{2} < 1$)

$\therefore$ not absolutely convergent.

<u>conditional convergence:</u>

$$\sum_{n=1871}^{\infty} \frac{(-1)^n}{n-1} = \sum_{n=1871}^{\infty} (-1)^n b_n$$

(i) $b_{n+1} \overset{?}{\leq} b_n$

$\frac{1}{n} \overset{?}{\leq} \frac{1}{n-1}$

$n-1 \overset{?}{\leq} n$

$-1 \leq 0$

ii) $\lim_{n \to \infty} b_n = 0$

$\therefore$ converges by alternating series test.

$\therefore$ conditionally convergent.

Bonus Question.

a. *(1 mark)* State the $K(\varepsilon)$ definition of the limit of a sequence.

b. *(4 marks)* Use the $K(\varepsilon)$ definition of the limit to prove the squeeze theorem.

a) $\forall \varepsilon > 0 \quad \exists K(\varepsilon) \quad s.t. \quad |a_n - L| < \varepsilon \quad \forall n \geq K(\varepsilon)$

where a_n is the sequence and L the limit of the sequence.

b) If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then

$\lim_{n\to\infty} b_n = L$

proof: Given $\varepsilon > 0$

$\exists K_a(\varepsilon) \quad s.t. \quad |a_n - L| < \varepsilon \quad \forall n \geq K_a(\varepsilon)$

$\exists K_c(\varepsilon) \quad s.t. \quad |c_n - L| < \varepsilon \quad \forall n \geq K_c(\varepsilon)$

Let $K(\varepsilon) = \max\{K_a(\varepsilon), K_c(\varepsilon)\}$

$$a_n \leq b_n \leq c_n$$

$$-\varepsilon < a_n - L \leq b_n - L \leq c_n - L < \varepsilon \qquad \forall n \geq K(\varepsilon)$$

$$-\varepsilon < b_n - L < \varepsilon$$

$$|b_n - L| < \varepsilon$$

$\therefore \lim_{n\to\infty} b_n = L$