Dawson College: **Calculus II**: 201-NYB-05 February 27, 2013

Last Name: SOLUTIONS

First Name: ____________________

Student ID: ____________________

Test 1

The time allowed for this test is 1 hour and 45 minutes. Please answer all questions in the space provided. Remember to write clearly and use correct notation.

Formulae

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Question 1. *(5 marks)* Evaluate the following definite integral using the limit definition (Riemann sums):

$$\int_{-1}^{3}(-2x^2+3x-1)dx$$

• $\Delta x = \frac{b-a}{n} = \frac{3-(-1)}{n} = \frac{4}{n}$ • $x_i = a + i\Delta x = -1 + \frac{4i}{n}$

• $f(x_i) = -2\left(-1+\frac{4i}{n}\right)^2 + 3\left(-1+\frac{4i}{n}\right) - 1 = -2 + \frac{16i}{n} - \frac{32i^2}{n^2} - 3 + \frac{12i}{n} - 1$

$= -\frac{32i^2}{n^2} + \frac{28i}{n} - 6$ • $f(x_i)\Delta x = \frac{-128i^2}{n^3} + \frac{112i}{n^2} - \frac{24}{n}$

• $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[\frac{-128i^2}{n^3} + \frac{112i}{n^2} - \frac{24}{n}\right] = \frac{-128}{n^3}\sum_{i=1}^{n} i^2 + \frac{112}{n^2}\sum_{i=1}^{n} i - \frac{24}{n}\sum_{i=1}^{n} 1$

$= \frac{-128}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{112}{n^2} \cdot \frac{n(n+1)}{2} - \frac{24}{n} \cdot n$

$\therefore \int_{-1}^{3}(-2x^3+3x-1)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[\frac{-128}{6} \cdot \frac{2n^2+3n+1}{n^2} + \frac{112}{2} \cdot \frac{n+1}{n} - 24\right]$

$= -\frac{64}{3} \cdot (2) + 56(1) - 24 = \frac{-32}{3}$

Question 2. *(5 marks)* Use the fundamental theorem of calculus part 2 (evaluation theorem) to find the following limit. Make sure to clearly show your work.

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\frac{\pi i}{4n}\right) \cdot \frac{\pi}{4n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

WHERE $x_i = \frac{\pi i}{4n}$ AND $\Delta x = \frac{\pi}{4n}$ $\therefore f(x) = \tan x$

SINCE $x_i = a + i\Delta x = a + i\left(\frac{\pi}{4n}\right) = \frac{\pi i}{4n} \Rightarrow a = 0$

$\Delta x = \frac{b-a}{n} = \frac{b-0}{n} = \frac{b}{n} = \frac{\pi}{4n}$ $\therefore b = \frac{\pi}{4}$

SO $L = \int_0^{\pi/4} \tan x\,dx = \Big[-\ln|\cos x|\Big]_0^{\pi/4}$

$= \left(-\ln\left|\cos\frac{\pi}{4}\right|\right) - \left(-\ln|\cos(0)|\right)$

$= -\ln\left(\sqrt{2}/2\right) + \ln|1|$

$= -\ln\left(\sqrt{2}/2\right)$

Question 3. Evaluate the following integrals:

(a) *(5 marks)* $\int_1^2 \frac{(x-1)^3}{x^2} dx$ $= \int_1^2 \frac{x^3 - 3x^2 + 3x - 1}{x^2} dx$

$= \int_1^2 \left(x - 3 + \frac{3}{x} - x^{-2}\right) dx = \left[\frac{x^2}{2} - 3x + 3\ln|x| + \frac{1}{x}\right]_1^2$

$= \left[\frac{(2)^2}{2} - 3(2) + 3\ln 2 + \frac{1}{2}\right] - \left[\frac{(1)^2}{2} - 3(1) + 3\ln(1) + \frac{1}{1}\right]$

$= 2 - 6 + 3\ln 2 + 1/2 - \frac{1}{2} + 3 - 0 - 1$

$= -2 + 3\ln 2$

(b) *(5 marks)* $\int \arctan(1/x) dx$

LET $u = \arctan(1/x)$ | $dv = dx$

$du = \frac{1}{1 + (1/x)^2} \cdot (-x^{-2}) dx$ | $v = x$

$= \frac{-1}{x^2+1} dx$

$= uv - \int v du = x\arctan(1/x) + \int \frac{x}{x^2+1} dx$

LET $u = x^2 + 1$

$du = 2x dx$

$\frac{du}{2x} = dx$

$= x\arctan(1/x) + \int \frac{x}{u} \frac{du}{2x}$

$= x\arctan(1/x) + \frac{1}{2}\int \frac{1}{u} du$

$= x\arctan(1/x) + \frac{1}{2}\ln|u| + C$

$= x\arctan(1/x) + \frac{\ln|x^2+1|}{2} + C$

(c) *(5 marks)* $\int_{-\pi/7}^{\pi/7} \frac{x^3 - x\cos x}{|x+\sin x|+3}\,dx = I$

LET $f(x) = \dfrac{x^3 - x\cos x}{|x+\sin x|+3} \Rightarrow f(-x) = \dfrac{(-x)^3 - (-x)\cos(-x)}{|(-x)+\sin(-x)|+3}$

$$= \frac{-x^3 + x\cos x}{|-x-\sin x|+3}$$

$$= -\frac{x^3 - x\cos x}{|x+\sin x|+3} = -f(x)$$

$\therefore$ f IS ODD

$\therefore \quad \underline{I = 0}$

(c) *(5 marks)* $\int_{\frac{1}{2}}^{\frac{e^4}{2}} \frac{\sqrt{\ln 2x}}{x}\,dx$

LET $u = \ln 2x$

$du = \frac{1}{2x} \cdot 2\,dx = \frac{1}{x}\,dx$

$x\,du = dx$

IF $x = \frac{1}{2} \Rightarrow u = \ln 2 \cdot \frac{1}{2} = 0$

$x = \frac{e^4}{2} \Rightarrow u = \ln 2\left(\frac{e^4}{2}\right)$
$= 4$

$$= \int_0^4 \frac{\sqrt{u}}{x} \cdot x\,du$$

$$= \int_0^4 u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_0^4$$

$$= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3}$$

Question 4. *(5 marks)* Use the midpoint rule with $n = 2$ to approximate the average value of $f(x) = e^{x^2}$ on the interval $[0,2]$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2-0}\int_0^2 e^{x^2}\,dx = \frac{1}{2}\int_0^2 e^{x^2}\,dx$$

$$\text{LET } I = \int_0^2 e^{x^2}\,dx \approx \sum_{i=1}^{2} f(\bar{x}_i)\Delta x$$

$$\Delta x = \frac{2-0}{2} = 1 \qquad \therefore \ \bar{x}_1 = 1/2,\ \bar{x}_2 = 3/2$$

$$\text{SO} \quad I \approx f(1/2)\cdot 1 + f(3/2)\cdot 1$$

$$= e^{(1/2)^2} + e^{(3/2)^2}$$

$$\simeq 10.5522303$$

$$\therefore \ f_{ave} \approx \frac{1}{2}(10.5522303)$$

$$\approx 5.386$$

Question 5. *(5 marks)* Find

$$\frac{d}{dx}\left[\int_{\sec 2x}^{3x^2} \frac{t^2}{\sqrt{t^2+1}}dt\right] = \frac{d}{dx}\left[-\int_0^{\sec 2x} \frac{t^2}{\sqrt{t^2+1}}dt + \int_0^{3x^2} \frac{t^2}{\sqrt{t^2+1}}dt\right] = D$$

LET $g(x) = \int_0^{x} \frac{t^2}{\sqrt{t^2+1}}dt$

$g'(x) = \frac{x^2}{\sqrt{x^2+1}}$ BY FTC 1

SO $\frac{d}{dx}\left[\int_0^{\sec 2x} \frac{t^2}{\sqrt{t^2+1}}dt\right] = \frac{d}{dx}\left[g(\sec 2x)\right] = g'(\sec 2x)\cdot\frac{d}{dx}\left[\sec 2x\right]$

$$= \frac{\sec^2 2x}{\sqrt{\sec^2 x+1}} \cdot \sec 2x \tan 2x \cdot 2$$

$$\frac{d}{dx}\left[\int_0^{3x^2} \frac{t^2}{\sqrt{t^2+1}}dx\right] = \frac{d}{dx}\left[g(3x^2)\right] = g'(3x^2)\cdot\frac{d}{dx}\left[3x^2\right]$$

$$= \frac{(3x^2)^2}{\sqrt{(3x^2)^2+1}} \cdot 6x$$

$$\therefore\ D = -\frac{\sec^2 2x}{\sqrt{\sec^2 x+1}}\cdot \sec 2x \tan 2x \cdot 2 + \frac{(3x^2)^2}{\sqrt{(3x^2)^2+1}}\cdot 6x$$

Question 6. *(5 marks)* Use integration by parts to evaluate

$$\int \sin(\ln x)dx$$

LET $u = \sin(\ln x)$ $dv = dx$

$du = \cos(\ln x) \cdot \frac{1}{x} dx$ $v = x$

$$= uv - \int v du = x\sin(\ln x) - \int x \cos(\ln x) \frac{1}{x} dx$$

$$= x\sin(\ln x) - \int \cos(\ln x) dx$$

LET $U = \cos(\ln x)$ $dV = dx$

$dU = -\sin(\ln x) \cdot \frac{1}{x} dx$ $V = x$

$$= x\sin(\ln x) - \left[UV - \int V dU\right]$$

$$= x\sin(\ln x) - \left[x\cos(\ln x) - \int x\left(-\sin(\ln x) \cdot \frac{1}{x}\right)dx\right]$$

$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$$

$$\therefore \quad 2\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) + C_1$$

$$\int \sin(\ln x)dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C$$

Question 7. *(5 marks)* Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$ and f'' is continuous. Find the value of

$$\int_1^4 xf''(x)dx$$

LET $u = x$ $\quad dv = f''(x)dx$

$du = dx$ $\quad v = f'(x)$

$$= uv\Big|_1^4 - \int_1^4 v\,du$$

$$= xf'(x)\Big|_1^4 - \int_1^4 f'(x)\,dx$$

$$= 4 \cdot f'(4) - (1) \cdot f'(1) - f(x)\Big|_1^4$$

$$= (4)f'(4) - (1)f'(1) - \big(f(4) - f(1)\big)$$

$$= 4(3) - (1)(5) - (7) + (2)$$

$$= 2$$

Bonus. *(3 marks)* The Fresnel function is defined by

$$S(x) = \int_0^x \sin(\pi t^2/2)dt$$

At what values of x does this function have local maximum values?

$S'(x) = \sin\left(\frac{\pi x^2}{2}\right)$ BY FTC1

$\therefore$ $S'(x) = 0 \iff \frac{\pi x^2}{2} = k\pi$ WHERE $k \in \mathbb{Z}$, $k \geq 0$

$\therefore$ $x^2 = 2k \Rightarrow x = \pm\sqrt{2k}$

$S''(x) = \cos\left(\frac{\pi x^2}{2}\right) \cdot (\pi x)$

IF $\underline{x = \sqrt{2k}}$, $S''(\sqrt{2k}) = \cos\left(\frac{\pi \cdot 2k}{2}\right) \cdot \left(\pi\sqrt{2k}\right)$

IF k IS EVEN $\cos(k\pi) = 1 > 0 \Rightarrow S''(\sqrt{2k}) > 0$ (LOCAL MIN)

k IS ODD $\cos(k\pi) = -1 < 0 \Rightarrow S''(\sqrt{2k}) < 0$ (LOCAL MAX)

IF $\underline{x = -\sqrt{2k}}$, $S''(-\sqrt{2k}) = \cos\left(\frac{\pi \cdot 2k}{2}\right) \cdot \left(-\pi\sqrt{2k}\right)$

IF k IS EVEN $\cos(k\pi) = 1 > 0 \Rightarrow S''(-\sqrt{2k}) < 0$ (LOCAL MAX)

IF k IS ODD $\cos(k\pi) = -1 < 0 \Rightarrow S''(-\sqrt{2k}) > 0$ (LOCAL MIN)

$\therefore$ $S(x)$ HAS LOCAL MAXIMUM VALUES AT

$x = \sqrt{2(2n+1)}$ AND $-\sqrt{2(2n)}$ WHERE $n \in \mathbb{Z}$, $n \geq 0$