Dawson College: **Calculus II**: 201-NYB-05

April 3, 2013

Last Name: SOLUTIONS

First Name: ____________

Student ID: ____________

Test 2

The time allowed for this test is 1 hour and 45 minutes. Please answer all questions in the space provided. Remember to write clearly and use correct notation.

Question 1. *(4 marks)*Write the partial fraction decomposition of the following fraction. You do not need to solve for the constants (A, B, C,...).

$$\frac{6x^2-3x-2}{x^2(x+3)(x^2+x+1)^2(x^2+4x+3)^2}$$

$$\boxed{\begin{array}{l} \underline{x^2+x+1} \\ b^2-4ac = 1^2-4(1)(1) < 0 \\ \text{IRREDUCIBLE.} \\ x^2+4x+3 = (x+1)(x+3) \end{array}}$$

$$= \frac{6x^2-3x-2}{x^2(x+3)^3(x^2+x+1)^2(x+1)^2}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3} + \frac{F}{x+1} + \frac{G}{(x+1)^2} + \frac{Hx+I}{x^2+x+1} + \frac{Jx+K}{(x^2+x+1)^2}$$

Question 2. *(5 marks)* Evaluate the following integral:

$$\int \frac{\sec^6(e^{-3x})}{e^{3x}}\,dx$$

LET $u = e^{-3x}$

$du = -3e^{-3x}dx$

$\dfrac{e^{3x}du}{-3} = dx$

$= -\frac{1}{3}\int \sec^6 u\,du$

$= -\frac{1}{3}\int (\sec^2 u)\sec^2 u\,du$

LET $t = \tan u$

$dt = \sec^2 u\,du$

$= -\frac{1}{3}\int (\tan^2 u + 1)^2 \sec^2 u\,du$

$= -\frac{1}{3}\int (t^2+1)^2\,dt$

$= -\frac{1}{3}\int (t^4 + 2t^2 + 1)\,dt$

$= -\frac{1}{3}\left[\frac{1}{5}t^5 + \frac{2}{3}t^3 + t\right] + c$

$= -\frac{1}{15}\tan^5(e^{-3x}) - \frac{2}{9}\tan^3(e^{-3x}) - \frac{1}{3}\tan(e^{-3x}) + c$

Question 3. *(5 marks)* Evaluate the following integral:

$$\int \frac{5x^4+2x^3+10x^2+x+3}{x^3+x}\,dx \quad = I$$

$$5x + 2$$

$$x^3+0x^2+x \,\overline{)\, 5x^4+2x^3+10x^2+x+3}$$

$$-(5x^4+0x^3+5x^2)$$

$$2x^3+5x^2+x$$

$$-(2x^3+0x^2+2x)$$

$$5x^2-x+3$$

$$\therefore \quad I = \int 5x+2 + \frac{5x^2-x+3}{x^3+x}\,dx$$

$$5x^2-x+3 = A(x^2+1)+(Bx+C)x = Ax^2+A+Bx^2+Cx$$

$$= (A+B)x^2 + Cx + A$$

$$\therefore \ A=3, \quad C=-1, \quad B=2$$

$$\therefore \quad I = \int 5x+2 + \frac{3}{x} + \frac{2x}{x^2+1} - \frac{1}{x^2+1}\,dx$$

$$= \frac{5}{2}x^2 + 2x + 3\ln|x| + \ln|x^2+1| - \arctan x + C$$

Question 4. *(5 marks)* Evaluate the following integral:

$$\int \frac{x+3}{x^2+2x+6}\,dx = \int \frac{x+1}{x^2+2x+6}\,dx + 2\int \frac{1}{x^2+2x+6}\,dx$$

$$\boxed{\begin{aligned} &\text{LET } u = x^2+2x+6 \\ &du = (2x+2)dx \\ &\quad = 2(x+1)dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{1}{u}\,du + 2\int \frac{1}{(x+1)^2+5}\,dx$$

$$\boxed{\begin{aligned} &x^2+2x+6 \\ &= x^2+2x+1 + 6 - 1 \\ &= (x+1)^2+5 \end{aligned}}$$

$$= \frac{1}{2}\ln|u| + \frac{2}{5}\int \frac{1}{\left(\frac{x+1}{\sqrt{5}}\right)^2+1}\,dx$$

$$= \frac{1}{2}\ln|x^2+2x+6| + \frac{2\sqrt{5}}{5}\arctan\left(\frac{x+1}{\sqrt{5}}\right)^2 + C$$

Question 5. *(5 marks)* Evaluate the following integral:

$$\int \frac{dx}{x^2\sqrt{x^2-16}}\,dx$$

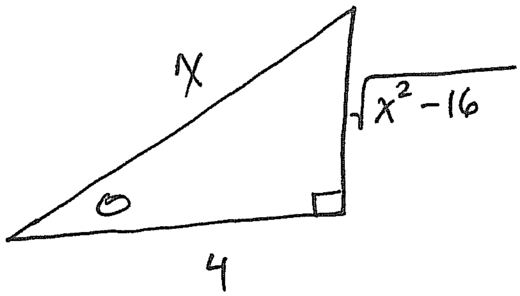
Let $x = 4\sec\theta$ on $0 \le \theta < \pi/2,$
$\pi \le \theta < 3\pi/2$

$dx = 4\sec\theta\tan\theta\,d\theta$

$$= \int \frac{4\tan\theta\sec\theta\,d\theta}{16\sec^2\theta \cdot 4\sqrt{\sec^2\theta - 1}}$$

since $\tan\theta \geq 0$ on specified interval

$$= \frac{1}{16}\int \frac{\tan\theta\sec\theta}{\sec^2\theta\tan\theta}\,d\theta = \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int \cos\theta\,d\theta$$



$$= \frac{1}{16}\sin\theta + C$$

$$= \frac{1}{16}\frac{\sqrt{x^2-16}}{x} + C$$

Question 6 . *(5 marks)* Determine whether the following improper integral converges or diverges. If it converges, find what it converges to.

$$\int_{-\infty}^{0} (5x+2)e^{3x}\,dx = \lim_{t \to -\infty} \int_{t}^{0} (5x+2)e^{3x}\,dx = I$$

$$\boxed{\begin{array}{ll} \text{LET } u = 5x+2 & dv = e^{3x}dx \\ du = 5dx & v = \frac{e^{3x}}{3} \end{array}}$$

$$\int (5x+2)\,e^{3x}\,dx$$

$$= \frac{5x+2}{3}e^{3x} - \frac{5}{9}\int e^{3x}\,dx$$

$$= \frac{5x+2}{3}e^{3x} - \frac{5}{9}e^{3x} + C$$

$$\therefore\ I = \lim_{t \to -\infty}\left[\frac{5x+2}{3}e^{3x} - \frac{5}{9}e^{3x}\right]_t^0$$

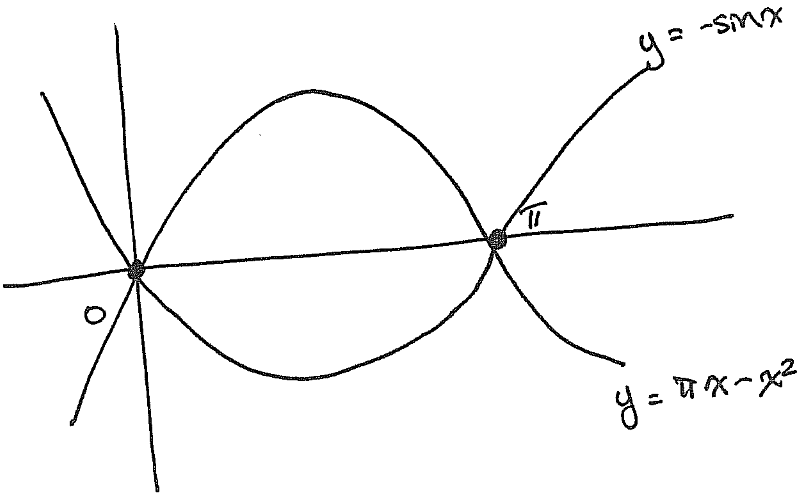
$$= \lim_{t \to -\infty}\left[\frac{2}{3} - \frac{5}{9} - \frac{5t+2}{3}e^{3t} + \frac{5}{9}e^{3t}\right]$$

$$\lim_{t \to -\infty} \frac{5t+2}{3}e^{3t} = \text{I.F. } \infty \cdot 0 = \lim_{t \to -\infty} \frac{5t+2}{3e^{-3t}} = \text{I.F. } \frac{\infty}{\infty}$$

$$\stackrel{\text{H}}{=} \lim_{t \to -\infty} \frac{5}{-9e^{-3t}} = \lim_{t \to -\infty} -\frac{5}{9}e^{3t} = 0$$

$$\therefore\ I = \frac{2}{3} - \frac{5}{9} - 0 - 0 = \frac{1}{9}$$

Question 8. *(5 marks)* Sketch the region bounded between $y = -\sin x$ and $y = \pi x - x^2$. Find the area of this region.



$$A = \int_0^{\pi} (\pi x - x^2) - (-\sin x)\, dx$$

$$= \left[\frac{\pi}{2} x^2 - \frac{1}{3} x^3 - \cos x \right]_0^{\pi}$$

$$= \frac{\pi}{2} (\pi)^2 - \frac{1}{3} (\pi)^3 - \cos \pi \quad - \quad 0 + 0 + \cos(0)$$

$$= 2 + \frac{\pi^3}{2} - \frac{\pi^3}{3} = 2 + \frac{\pi^3}{6} \text{ units}^2$$

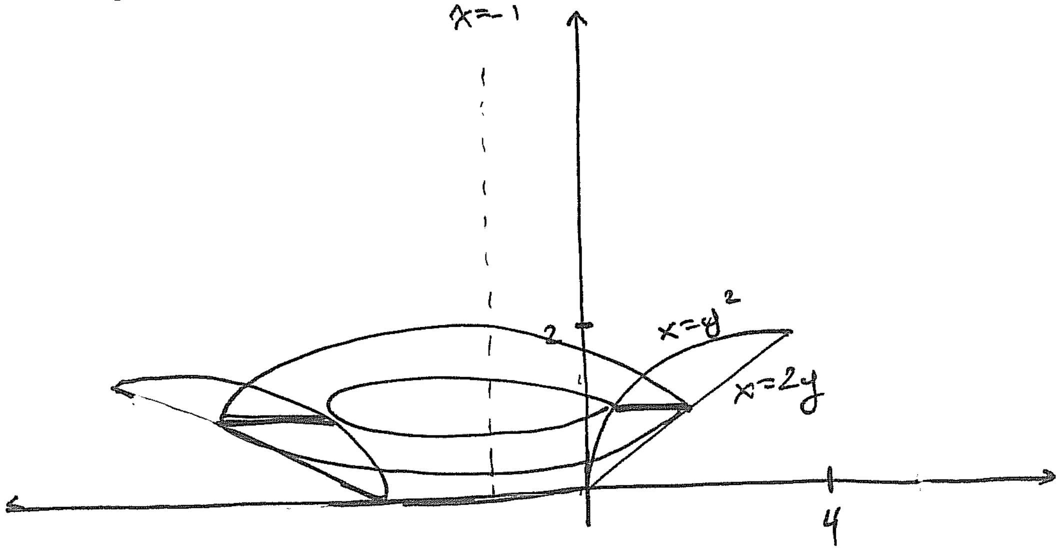
Question 9. *(5 marks)* Find the volume (using the circle/washer method from section 7.2) of the solid obtained by rotating the region bounded $y = \sqrt{x}$ and $y = x/2$ about the the line $x = -1$. Make sure to do a rough sketch of this region.

INTERSECTION $\sqrt{x} = x/2$

$$x = \frac{x^2}{4}$$

$$0 = x^2 - 4x$$

$$x = 0, 4$$



WASHER!

$$A(y) = \pi R^2 - \pi r^2$$

$$= \pi(1+2y)^2 - \pi(1+y^2)^2$$

$$= \pi\left[(4y^2+4y+1) - (1+2y^2+y^4)\right]$$

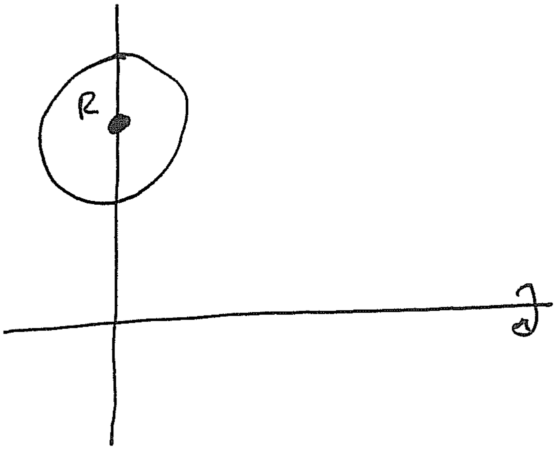
$$= \pi\left(-y^4 + 2y^2 + 4y\right)$$

$$V = \pi \int_0^2 -y^4 + 2y^2 + 4y \, dy$$

$$= \pi\left[-\frac{y^5}{5} + \frac{2}{3}y^3 + 2y^2\right]_0^2 = \pi\left[-\frac{32}{5} + \frac{16}{3} + 8 - 0 - 0 - 0\right]$$

$$= \frac{104}{15}\pi \text{ units}^3$$

Bonus. *(3 marks)* Set up an integral for the volume of the solid torus. Interpret this integral as an area to evaluate it.



$$x^2+(y-R)^2=r^2$$

$$y = R \pm \sqrt{r^2-x^2}$$

$$A(x) = \pi R^2 - \pi r^2 = \pi\left[R+\sqrt{r^2-x^2}\right]^2 - \pi\left[R-\sqrt{r^2-x^2}\right]^2$$

$$= \pi\left[R^2+2R\sqrt{r^2-x^2} + r^2-x^2\right] - \pi\left[R^2-2R\sqrt{r^2-x^2} + (r^2-x^2)\right]$$

$$= 4R\sqrt{r^2-x^2}$$

$$\therefore \; V = \int_{-r}^{r} \pi 4R\sqrt{r^2-x^2}\,dx = 4\pi R \int_{-r}^{r} \underbrace{\sqrt{r^2-x^2}\,dx}_{\text{UPPER HALF CIRCLE WITH RADIUS } r}$$

$$= 4R\pi \cdot \frac{1}{2}\pi r^2$$

$$= 2\pi^2 R r^2 \text{ units}^3$$