Dawson College: **Calculus II**: 201-NYB-05

May 1, 2013

Last Name: SOLUTIONS

First Name: ____________

Student ID: ____________

Test 3

The time allowed for this test is 1 hour and 45 minutes. Please answer all questions in the space provided. Remember to write clearly and use correct notation.

Question 1. *(1 mark each)* Put a T or F next to each statement to indicate whether it is true or false.

(a) $\lim_{n\to\infty} a_n \neq 0$ means that $\sum_{i=1}^{\infty} a_n$ diverges. T

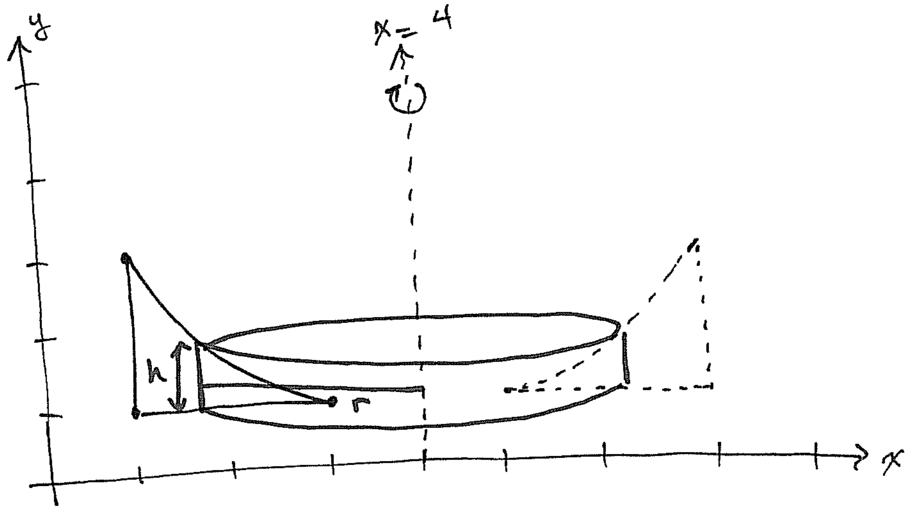
(b) If $\sum_{i=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$. T

(c) If the sequence of partial sums has limit $\lim_{n\to\infty} s_n \neq 0$ then its series $\sum_{i=1}^{\infty} a_n$ diverges. F

(d) If the series $\sum_{i=1}^{\infty} |a_n|$ is convergent it is possible that the series $\sum_{i=1}^{\infty} a_n$ is divergent. F

(e) If the series $\sum_{i=1}^{\infty} a_n$ is conditionally convergent then the series $\sum_{i=1}^{\infty} |a_n|$ is divergent. T

Question 2. *(5 marks)* Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curves $y = 3/x$, $y = 1$ and $x = 1$ about the line $x = 4$. (Remember to sketch this region.)



$A(x) = 2\pi r h$, $\quad r = 4 - x$, $\quad h = \frac{3}{x} - 1$

$$V = \int_1^3 2\pi(4-x)\left(\frac{3}{x}-1\right)dx = 2\pi\int_1^3\left(\frac{12}{x} - 7 + x\right)dx$$

$$= 2\pi\left[12\ln x - 7x + \frac{x^2}{2}\right]_1^3$$

$$= 2\pi\left[12\ln 3 - 7(3) + \frac{(3)^2}{2} + 12\ln(1) + 7(1) - \frac{1}{2}\right]$$

$$= 2\pi\left[12\ln 3 - 10\right] \text{ units}^3$$

Question 3. *(5 marks)* Find the exact length of the curve $x = \ln(\sin y)$ on $\frac{\pi}{4} \le y \le \frac{\pi}{3}$

$$\frac{dx}{dy} = \frac{1}{\sin y} \cdot \cos y = \cot y \Rightarrow 1 + \cot^2 y = \csc^2 y$$

$$\therefore \quad L = \int_{\pi/4}^{\pi/3} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{\pi/4}^{\pi/3} \sqrt{\csc^2 y}\, dy = \int_{\pi/4}^{\pi/3} \csc y\, dy$$

SINCE $\csc y \geq 0$ ON $\pi/4 \le y \le \pi/3$

$$= \Big[-\ln|\csc x + \cot x| \Big]_{\pi/4}^{\pi/3}$$

$$= -\ln\left|\csc\left(\frac{\pi}{3}\right) + \cot(\pi/3)\right| + \ln\left|\csc\left(\frac{\pi}{4}\right) + \cot\left(\frac{\pi}{4}\right)\right|$$

$$= -\ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| + \ln\left|\frac{2}{\sqrt{2}} + 1\right|$$

$$= -\ln(\sqrt{3}) + \ln\left(\frac{2+\sqrt{2}}{2}\right)$$

Question 4. *(5 marks)*
(a) Show that the sequence

$$a_n = \frac{\sin^2 n - 2\cos^2 n}{e^n}$$

is bounded but not monotonic.

$$-2 \le \sin^2 n - 2\cos^2 n \le 1$$

$$\Rightarrow \quad \frac{-2}{e^n} \le \frac{\sin^2 n - 2\cos^2 n}{e^n} \le \frac{1}{e^n} \quad \Rightarrow \quad \frac{-2}{e} \le \frac{-2}{e^n} \le \frac{\sin^2 n - 2\cos^2 n}{e^n} \le \frac{1}{e^n} \le \frac{1}{e}$$

$\therefore$ SEQUENCE IS BOUNDED.

$$a_1 = \frac{\sin^2 1 - 2\cos^2 1}{e} \approx 0.045698 \qquad a_2 = \frac{\sin^2 2 - 2\cos^2 2}{e^2} \approx 0.06502$$

$$a_3 = \frac{\sin^2(3) - 2\cos^2(3)}{e^3} \approx -0.0966$$

SO $a_1 < a_2$ BUT $a_2 > a_3$ SO NOT MONOTONIC

(b) Use the squeeze theroem to show that this sequence converges. Find its limit.

$$\frac{-2}{e^n} \le \frac{\sin^2 n - 2\cos^2 n}{e^n} \le \frac{1}{e^n}$$

$$\lim_{n\to\infty}\left(-\frac{2}{e^n}\right) = 0 \quad , \quad \lim_{n\to\infty}\left(\frac{1}{e^n}\right) = 0$$

SO BY SQUEEZE THEOREM

$$\lim_{n\to\infty} \frac{\sin^2 n - 2\cos^2 n}{e^n} = 0$$

Question 5. Determine if the series is convergent or divergent. If it is convergent, find its sum.

(a) *(5 marks)* $\sum_{n=2}^{\infty} \frac{2(-3)^n+4}{5^{n+1}} = \sum_{n=2}^{\infty} \frac{2(-3)^n}{5^{n+1}} + \sum_{n=2}^{\infty} \frac{4}{5^{n+1}}$ AS LONG AS BOTH SERIES CONVERGE

$$= \sum_{n=1}^{\infty} \frac{2}{5}\left(\frac{-3}{5}\right)^n - a_1 + \sum_{n=1}^{\infty} \frac{4}{5}\cdot\left(\frac{1}{5}\right)^n - a_1'$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{-6}{25}\left(\frac{-3}{5}\right)^{n-1}}_{\text{GEOMETRIC } |r| = \frac{3}{5} < 1 \ \therefore \text{ CONVERGES}} - \frac{2}{5}\left(\frac{-3}{5}\right)^1 + \underbrace{\sum_{n=1}^{\infty} \frac{4}{25}\left(\frac{1}{5}\right)^{n-1}}_{\text{GEOMETRIC } |r| = \frac{1}{5} < 1 \ \therefore \text{ CONVERGES}} - \frac{4}{25}$$

$$= \frac{-6/25}{1-(-3/5)} + \frac{6}{25} + \frac{4/25}{1-1/5} - \frac{4}{25}$$

$$= \frac{-6}{25}\cdot\frac{5}{8} + \frac{6}{25} + \frac{4}{25}\cdot\frac{5}{4} - \frac{4}{25}$$

$$= -\frac{3}{20} + \frac{6}{25} + \frac{1}{5} - \frac{4}{25}$$

$$= \frac{13}{100}$$ (SERIES CONVERGES)

(b) *(5 marks)* $\sum_{n=2}^{\infty} \frac{e^n}{\ln(n)}$

$$\lim_{n\to\infty} \frac{e^n}{\ln(n)} = \text{I.F. } \frac{\infty}{\infty} \overset{H}{=} \lim_{n\to\infty} \frac{e^n}{1/n} = \lim_{n\to\infty} ne^n = \infty$$

$\therefore$ THE SERIES DIVERGES BY TEST FOR DIVERGENCE.

(c) *(5 marks)* $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1} = S$

$$\frac{2}{n^2-1} = \frac{A}{n+1} + \frac{B}{n-1}$$

$2 = A(n-1) + B(n+1)$ IF $n = 1 \Rightarrow 1 = B$, IF $n = -1 \Rightarrow A = -1$

$$\therefore \quad S = \sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$S_n = \left(\frac{1}{1} - \cancel{\frac{1}{3}} \right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} \right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} \right)$$

$$+ \ldots + \left(\cancel{\frac{1}{n-4}} - \cancel{\frac{1}{n+2}} \right) + \left(\cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n+1}} \right) + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n+2} \right)$$

$$+ \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1} \right)$$

$$= 1 + \frac{1}{2} - \frac{1}{n+2} - \frac{1}{n+1}$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{2} - \frac{1}{n+2} - \frac{1}{n+1} \right) = \frac{3}{2} - 0 - 0 = \frac{3}{2}$$

$\therefore$ THE SERIES CONVERGES AND

$$S = \frac{3}{2}$$

Question 6. *(5 marks)* Determine if the following series converges or diverges.

$$\sum_{i=9}^{\infty} \frac{3}{n(\ln n)^2}$$

LET $f(x) = \dfrac{3}{x(\ln x)^2}$

THEN f IS POSITIVE AND CONTINUOUS ON $[e, \infty)$ $([9, \infty))$

SINCE x AND $\ln x$ ARE.

$$f'(x) = -\frac{\left[1 \cdot (\ln x)^2 + 2x(\ln x) \cdot \frac{1}{x}\right]}{\left[x(\ln x)^2\right]^2}$$

$$= \frac{-(\ln x)^2 - 2\ln x}{\left[x(\ln x)^2\right]^2} < 0 \quad \text{ON } [e, \infty)$$

$([9, \infty))$

$$\int_9^{\infty} \frac{3}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_9^t \frac{3}{x(\ln x)^2}\,dx = L$$

$$\int \frac{3}{x(\ln x)^2}\,dx = 3\int \frac{1}{u^2}\,du$$

$\boxed{\text{LET } u = \ln x \quad du = \frac{1}{x}dx}$

$$= 3\left(-\frac{1}{u}\right) + C = \frac{-3}{\ln x} + C$$

$$\therefore L = \lim_{t\to\infty} \left[\frac{-3}{\ln x}\right]_9^t = \lim_{t\to\infty} \left[\frac{-3}{\ln t} + \frac{3}{\ln 9}\right] = 0 + \frac{3}{\ln 9}$$ CONVERGES

$\therefore$ THE SERIES CONVERGES BY INTEGRAL TEST.

Question 7 . Determine if the series is absolutely convergent, conditionally convergent or divergent.

(a) *(5 marks)* $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}n^2}{(3n)!}}_{a_n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)^2\pi^{2n+2}}{(3n+3)!}\cdot\frac{(3n)!}{(-1)^{n+1}n^2\pi^{2n}}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\frac{\pi^2}{(3n+1)(3n+2)(3n+3)}$$

$$= \cdot\ 0 < 1$$

∴ THE SERIES CONVERGES ABSOLUTELY BY RATIO TEST.

(b) *(5 marks)* $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n (n)}{n^2+1}}_{a_n}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^n n}{n^2+1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

LET $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n}{n^2+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = \frac{1}{1+0} = 1 > 0$$

$\therefore$ SINCE $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES BY P-SERIES, $p = 1 \leq 1$

SO DOES $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ BY LIMIT COMPARISON TEST

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ LET $b_n = \frac{n}{n^2+1}$, $f(x) = \frac{x}{x^2+1}$

1) $f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$ FOR $x > 1$

$\therefore$ b_n IS DECREASING

2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n} = \frac{0}{1+0} = 0$

SO $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ CONVERGES BY ALTERNATING SERIES TEST

$\therefore$ $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ IS CONDITIONALLY CONVERGENT.

Bonus. *(2 marks)* For which x values is the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{\underbrace{2^{2n}(n!)^2}_{a_n}}$$

convergent?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,x^{2n+2}}{2^{2n+4}\,[(n+1)!]^2}\cdot\frac{2^{2n}\,(n!)^2}{(-1)^n\,x^{2n}}\right|$$

$$= \lim_{n\to\infty}\frac{x^2}{2^4\,(n+1)^2} = 0<1 \text{ FOR ALL } x \text{ VALUES}$$

$\therefore$ THE SERIES CONVERGES (ABSOLUTELY) FOR ALL VALUES OF x.