Improper Integrals

So far, we have solved the area problem for a continuous function f on the interval [a,b]. We've seen that we can find the net area by taking a definite integral, for example:

$$\int_1^2 \frac{1}{x^2}dx$$

Does it make sense to talk about the area on an unbounded interval? This would seem to correspond to a definite integral on an unbounded interval.

$$\int_1^\infty \frac{1}{x^2}dx$$

Let's consider the area under $1/x^2$ from 1 to some value t.

A(t)=

Now let's see what happens as b get's larger.

t	
A(t)	

Let's try the same thing with the function 1/x.

t	
A(t)	

We can see that when we are looking for $\int_a^{\infty} f(x)dx$ we should find

$$\lim_{t \to \infty} \int_a^t f(x)dx$$

Definition: An integral is called an **improper integral (of type I)** if one of the limits of integration is infinite:

a) If f(x) is continuous on $[a, \infty)$

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided this limit exists as a finite number.

b) If f(x) is continuous on $(-\infty, b]$

$$\int_{-\infty}^b f(x)dx = \lim_{t\to\infty} \int_t^b f(x)dx$$

provided this limit exists as a finite number.

c) If f(x) is continuous on $(-\infty, \infty)$

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

where c is any number.

If the limits exist we say that the integrals converge, otherwise we say that they diverge. **Both** inetgrals on the right in c) must converge for us to say that the integral on the left converges.

Examples: Evaluate the following improper integrals

1) $\displaystyle\int_1^{\infty} \frac{1}{x^2} dx$

2) $\displaystyle\int_2^{\infty} \frac{4}{\sqrt[4]{x}} dx$

3) $\displaystyle\int_0^{\infty} x e^{-x/2} dx$

4) $\int_{-\infty}^{\infty} xe^{x^2}dx$

5) $\int_{-\infty}^{\infty} \frac{5}{x^2 + 4x + 8} dx$

Remember, the fundamental theorem of calculus part 2 (evaluation theorem) requires the function be continuous. This means that we can't use the FTC2 directly on an integral that looks like

$$\int_0^6 \frac{4}{\sqrt{6-x}}dx$$

since the integrand is not continuous at 0. We would also call this an improper integral.

Improper integrals:

a)

b)

c)

Definition: An integral is called an **improper integral (of type II)** if the integrand has a discontinuity in the interval of integration.

a) If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

if this limit exists.

b) If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if this limit exists.

Again, if the limit exists we say the integral is convergent, otherwise we say it is divergent.

c) If f has a discontinuity at c and $a < c < b$ then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(Both integrals on the right must converge in order for the integral on the left to converge).

Examples: Find the following

1) $\displaystyle\int_0^6 \frac{4}{\sqrt{6-x}}dx$

2) $\displaystyle\int_0^4 \frac{8}{x}dx$

3) $\displaystyle\int_0^3 \frac{dx}{x-1}$