Partial Fractions:

Once the numerator has a smaller degree than the denominator, the next step is to factor the denominator as much as possible.

It can be shown that any polynomial $q(x)$ can be factored as a product of linear factors (of the form $ax + b$) and irreducible quadratic factors (of the form $ax^2 + bx + c$ where $b^2 - 4ac < 0$)

Important: all remaining quadratics must be irreducible!!! If $b^2 - 4ac \geq 0$ then we need to factor the quadratic.

Once $q(x)$ is factored the next step is to express the function as a sum of partial fractions of the form

or

To illustrate, recall that we can combine fractions

$$\frac{5}{x+3}-\frac{2}{x+1}$$

And so if we were asked to find

$$\int \frac{3x-1}{(x+3)(x+1)}dx$$

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There are four cases we will look at.

Case 1: The denominator q(x) is the product of distinct linear factors $ax + b$

$$q(x) = (a_1x + b_1)(a_2x + b_2) \ldots (a_nx + b_n)$$

For each linear factor we need one term of the type $\dfrac{A}{ax + b}$ so

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x + b} + \frac{A_2}{a_2x + b} + \cdots + \frac{A_n}{a_nx + b}$$

Example: Let's write $\dfrac{x + 5}{x^2 - 5x + 4}$ as a sum of simpler fractions.

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Example: Find

$$\int \frac{x+5}{x^2-5x+4}dx$$

Example: Evaluate

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$$

Case 2: The denominator q(x) is the product of linear factors $ax + b$ some of which are repeated.

For each $(ax+b)^n$ in the denominator we need

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\ldots+\frac{A_n}{(ax+b)^n}$$

For example if we wanted to write the following as a sum of simpler fractions

$$\frac{x^2+2x-3}{x^2(x+1)(2x+1)^3}$$

Example: Find

1) $\displaystyle\int \frac{x^2 - 2x - 5}{x^3 - 5x^2}dx$

2) $\displaystyle\int \frac{5x^2 - 9x}{(x-4)(x-1)^2}dx$

Case 3: q(x) also contains irreducible quadratic factors $ax^2 + bx + c$, where $b^2 - 4ac < 0$, none of which are repeated.

For each $ax^2 + bx + c$ we need a factor of the type

$$\frac{Ax + b}{ax^2 + bx + c}$$

Example: Find

1) $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$

2) $\displaystyle\int \frac{5x^2 + 12}{(x-2)(x^2+4)}dx$

Case 4: q(x) contains a repeated irreducible quadratic factor.

For each $(ax^2 + bx + c)^n, \;\; b - 4ac < 0$ we need

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \frac{A_3x + B_3}{(ax^2 + bx + c)^3} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

For example, if we wanted to write the following as the sum of simpler fractions we would need

$$\frac{3x^4 + 2x - 1}{x^2(3x - 2)(x^2 + x + 1)(x^2 + 1)^3}$$

Example: Evaluate

1) $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} dx$

2) $\displaystyle\int \frac{2x^4 - x + 1}{x(x^2+1)^2}dx$

Examples: Find the following

1) $\displaystyle\int \frac{2x^3+2x^2-95x+40}{x^3+x^2-20x}dx$

2) $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$

3) $\displaystyle\int \frac{x-3}{(x^2+2x+10)^2}dx$