Probability

The **sample space**, S, of an experiment is the set of all possible outcomes of that experiment

Examples:

1) Experiment: Tossing a coin and observing heads or tails.

2) Experiment: Tossing a coin and observing heads or tails three times.

3) Two gas stations are located across from each other and have four gas pumps each. The experiment is to observe how many pumps are in use at 4:00pm for both gas stations.

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An **event** is any collection (subset) of outcomes contained in the sample space S.

A **simple event** consists of only one outcome.

A **compound event** consists of more than one outcome.

Example: Returning to the gas stations, here are some events:

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Some Relations from Set Theory

1) The **union** of two events A and B denoted $A \cup B$ (read A or B) is the event consisting of all outcomes that are either

2) The **intersection** of two events A and B denoted $A \cap B$ (read A and B) is the event consisting of all outcomes that are both in A and in B.

3) The **Compliment** of an event A denoted A^c or A' is the set of all outcomes that are not in A.

Back to the gas pump example.

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When events A and B have no outcomes in common they are said to be **disjoint** or **mutually exclusive**.

This is written $A \cap B = \varnothing$ where $\varnothing$ is the null set or empty set, that is, the event consisting of no outcomes.

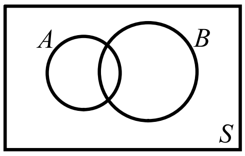
These ideas can be extended to multiple events $A_1, A_2, A_3, \ldots, A_n$

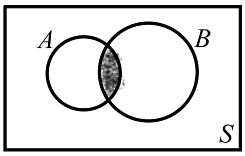
$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n =$ the set of outcomes that are contained in A_1 and A_2 and A_3 and $\ldots$ and A_n

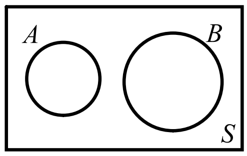
$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n =$ the set of outcomes that are contained in either $A_1, A_2, A_3, \ldots, \text{or } A_n$

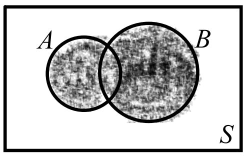
Venn Diagrams

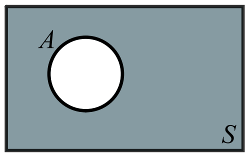
Venn Diagrams are illustrations that help us visualize events.











Example: Illustrate A^c, B^c and $A^c \cap B^c$

Probability

Given an experiment and sample space S, the objective is to assign to each event a number $P(A)$ **called the probability of the event A** which will give a precise measure of the chance A will occur.

We will look at two types of probability:

1) **Empirical probability** is the probability of event A as a result of experimentation.

empirical probability of A = $\frac{\text{number of times A occured}}{\text{number of trials}}$

$$P'(A) = \frac{n(A)}{n}$$

2) **Theoretical probability** is the probability of event A determined using the sample space of all possible outcomes.

theoretical probability of A = $\frac{\text{number of times A occurs in sample space}}{\text{number of elements in sample space}}$

$$P(A) = \frac{n(A)}{n(s)}$$

Basic Properties of Probability (Axioms)

The following three axioms must always hold when assigning probabilities

Axiom 1 For any event A, $P(A) \geq 0$

Axiom 2 $P(S) = 1$

Axiom 3 If $A_1, A_2, A_3 \ldots$ is an infinite collection collection of disjoint events ($A_i \cap A_j = \varnothing$ for all i, j where $i \neq j$) then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

What about $P(\varnothing)$? Intuitively the probability that no events occur should be 0. Will this be consistent with our three axioms?

Proposition: $P(\varnothing) = 0$

Proof:

Proposition: Axiom 3 also holds for a finite number of disjoint sets $A_1, A_2, A_3, \ldots, A_k$. That is

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) = \sum_{i=1}^{k} P(A_i) = P(A_1) + P(A_2) + \ldots + P(A_k)$$

Proof:

Consider the following experiment:

Toss thumbtack in the air and observe it land either point up (U) or point down (U).

sample space =

U and D are disjoint (because they cannot both occur at the same time) and $U \cup D = S$

But by axiom 2 $P(S) = 1$ so

Which would mean that

A possible probability assignment would be P(D) = 0.25 and P(U)=0.75. This would be consistent with the three axioms.

Another might be P(D)=0.5 and P(U)=0.5.

But how does one make such an assignment of probabilities?

For example, with the thumbtacks the appropriate assignment depends on the nature of the thumbtack and on one's interpretation of probability.

Since we don't know the sample space in this situation it makes sense to use

empirical probability.

If an experiment can be repeatedly performed in an identical and independent fashion then we said the empirical probability of event A is

where $n(A)$ is the number of replications of the experiment in which event A occurs and n is the total number of times the experiment is performed.

$\frac{n(A)}{n}$ is also called the **relative frequency** of occurrence of the event A.

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Example:

Empirical Evidence, based on the results of many of these sequences of repeatable experiments indicates that as n grows large, the relative frequency $\frac{n(A)}{n}$ stabilizes.

The value that $\frac{n(A)}{n}$ tends to as n grows arbitrarily large is called the **limiting relative frequency** of the event A.

The idea here is that

In our example above:

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When various outcomes of an experiment are equally likely (same probability assigned to each simple event) or when we know the sample space, it makes sense to use the theoretical probability.

Example: Tossing a fair six-sided die.

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