8.1 Sequences

A **sequence** is a function whose domain is the natural numbers. The usual notation for the nth term of a sequence is $a_n = f(n)$.

Usually

$$f(x) = \frac{1}{x}$$

function on the real numbers

$$f(n) = \frac{1}{n}$$

function on the natural numbers

For the function $f(n) = \frac{1}{n}$ we could make a chart

n	
$f(n)$	

Notice that a sequence can be thought of as a list of numbers written in a specific order

$$a_1, a_2, a_3, \ldots$$

In the case of the sequence above

Common notation for a sequence:

$$\{a_1, a_2, a_3, \ldots\}, \quad \{a_n\}, \quad \{a_n\}_{n=1}^{\infty}$$

Examples:

1) $\left\{\dfrac{n^2}{2n+1}\right\}_{n=1}^{\infty}$

2) $\left\{\dfrac{e^{n+1}}{\ln(n+2)}\right\}$

Sometimes only the first few terms of the sequence are given and not the general nth term. Sometimes we can find the general term by assuming the pattern continues.

Examples:

1) $\left\{\frac{3}{5}, \frac{4}{25}, \frac{5}{125}, \frac{6}{625}, \frac{7}{3125}, \dots\right\}$

2) $\left\{1, -1, \frac{3}{6}, -\frac{4}{24}, \frac{5}{120}, -\frac{6}{120}, \dots\right\}$

n factorial, denoted by $n!$, is defined by

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (n-1) \cdot n$$

where n is a positive integer and

$$0! = 1$$

For example

$$3! = 1 \cdot 2 \cdot 3 = 6$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

Sometimes we can't (at least very easily).

Example:

$$\{1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots\}$$

Since a sequence is a function we sketch its graph. For example, $f(n) = \dfrac{1}{n}$

We can see that it makes sense to talk about whether a sequence approaches a number or not as the values get larger and larger.

Definition: A sequence $\{a_n\}$ had the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty$$

if we can make the terms a_n as close to L as we like by making n sufficiently large. If $\lim_{n\to\infty} a_n$ exists (as a number) we say the sequence **converges**, otherwise the sequence diverges.

Graphical Interpretation

Examples: Find the limit of the following sequences:

1) $\left\{\dfrac{3n^2+5n-6}{7n^2-2n+5}\right\}$

2) $\displaystyle\lim_{n\to\infty}\frac{\sqrt{4n^2-n+1}}{\sqrt{9n^2+1}}$

At first glance it doesn't seem like we can use L'Hôpital's rule since sequences are not continuous functions (what would the derivative be of a bunch of points?).

The following theorem helps us out of this problem.

<u>Theorem</u> If $\lim_{n\to\infty} f(x) = L$ and $f(n) = a_n$ for all positive integers n then

$$\lim_{n\to\infty} a_n = L$$

Basically, this theorem tells us that if we swap out the n for x for a sequence, and this new function has a limit than so does the sequence. (What if the new function does not have a limit? We'll come back to this question.)

Examples:

1) $\lim_{n\to\infty} \frac{\ln(\ln n)}{n}$

2) $\lim_{n\to\infty} \left(\arctan\left(\frac{1}{n}\right)\right)^{1/n}$

Definition: $\lim_{n \to \infty} a_n = \infty$ (diverges to ∞) means for every positive number M there exists an integer k such that if $n > k$ then $a_n > M$.

So, what if the new function $f(x)$ does not have a limit? Does this mean that the sequence $a_n = f(n)$ does not have a limit as well?

Theorem: (Limit laws for sequences)

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant then

1) $\lim_{n\to\infty}(a_n + b_n) =$

2) $\lim_{n\to\infty}(a_n - b_n) =$

3) $\lim_{n\to\infty} ca_n =$

4) $\lim_{n\to\infty} a_n b_n =$

5) $\lim_{n\to\infty} \frac{a_n}{b_n} =$

Example: $\lim_{n\to\infty} \frac{4^n}{5^{n+2}}$

Theorem: If $\lim_{n \to \infty} |a_n| = 0$ then $\lim_{n \to \infty} a_n = 0$

Example: Find the limit of $a_n = \frac{(-1)^n n}{n^2 + 1}$ if it exists.

Theorem: (Squeeze Theorem)

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$ and $a_n \le b_n \le c_n$ for all n then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$$

Example: Determine if the sequence converges or diverges. If it converges find the limit.

$$b_n = \frac{(-1)^n \sin(2n) \arctan n}{n^2}$$

Theorem: (Uniqueness of Limits)

A sequence can have at most one limit.

Definition:

A sequence a_n is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$

$$a_1 < a_2 < a_3 < \ldots$$

A sequence a_n is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$

$$a_1 > a_2 > a_3 > \ldots$$

A sequence is called **monotonic** if it is either increasing or decreasing.

Example:

1) Show that $a_n = \left(\frac{3}{2}\right)^2$ is an increasing monotonic sequence.

2) Show that $a_n = -\ln n$ is an decreasing monotonic sequence.

Definition: A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

A sequence $\{a_n\}$ is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all } n \geq 1$$

A **bounded** sequence is bounded above and below.

Theorem: (Monotonic Sequence Theorem)

Every bounded, monotonic sequence is convergent.

Example: Use the monotonic sequence theorem to show that the sequence $a_n = \dfrac{2n-3}{3n+4}$ converges.