Taylor and Maclaurin Series

In the previous section we saw that

$$J_0 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \dots$$

is a function whose domain is $\mathbb{R}$.

Are there other functions that we can write as a power series? That is, given a function $f(x)$, can we find constants $c_0, c_1, c_2, c_3, \dots$ so that

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \dots = \sum_{n}^{\infty} c_n(x-a)^n \text{ ?}$$

Suppose it is possible. What would the constants be?

$f(a) =$

It turns out that the derivative of a power series is the term by term derivative, that is

$f'(x) =$

and so

$f'(a) =$

continuing in this way

Theorem: If f has a power series representation (expansion) at a, that is if

$$f(x) = \sum_{n}^{\infty} c_n (x-a)^n \qquad \text{on} \qquad |x-a| < R$$

then its coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

We see that we would get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$

This is called the **Taylor series of the function f at a**.

For the special case where $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots$$

which is called the **Maclaurin Series of** $f(x)$.

Remember, this is all assuming f is equal to its Taylor series expansion, there are functions that are not equal to their Taylor series.

Example: Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Example: Find the Taylor series for $f(x) = e^x$ at $a = 2$.

Example: Find the Maclaurin series for $\sin x$.

Example: Find the Taylor series for $f(x) = \ln x$ at $a = 2$.

Example: Find the Maclaurin series for $f(x) = (1+x)^k$ (where k is a real number).

Example: Find the Taylor series for $f(x) = x^{\frac{3}{4}}$ at $a = 1$.