<u>The Integral and Comparison Tests</u>

It turns out that there is a very close relationship between series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x)dx$ where $f(n) = a_n$ and f(x) is positive and decreasing.

Let's examine this relationship:

Theorem: The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:

1) If $\int_1^{\infty} f(x)dx$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

2) If $\int_1^{\infty} f(x)dx$ is divergent then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: It is not necessary to start the integral test at n=1. For example, to test $\sum_{n=4}^{\infty} \frac{1}{(n-2)^3}$ we use $\int_4^{\infty} \frac{1}{(x-2)^3} dx$

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges or converges.

Example: Is the series $\sum_{n=1}^{\infty} ne^{-n}$ convergent or divergent?

Example: For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent.

p-series

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

These series will be useful when we use the comparison tests.

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n then $\sum a_n$ is also convergent.

b) If $\sum b_n$ is convergent and $a_n \geq b_n$ for all n then $\sum a_n$ is also convergent.

Examples: Determine whether the following series converge or diverge:

1) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{5n^3 + 2n + 1}$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^{3/2} - 2\sqrt{n} - 1}$

3) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - 5}$

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Example: Determine whether the following series converge or diverge.

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - 5}$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$

3) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

4) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$