The Integral and Comparison Tests

It turns out that there is a very close relationship between series $\sum_{n=1}^{\infty} a_n$ and the improper integral $\int_1^{\infty} f(x) dx$ where $f(n) = a_n$ and f(x) is positive and decreasing.

Let's examine this relationship:

Theorem: The Integral Test

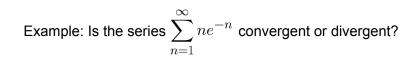
Suppose *f* is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x) dx$ is convergent. In other words:

1) If
$$\int_{1}^{\infty} f(x)dx$$
 is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.
2) If $\int_{1}^{\infty} f(x)dx$ is divergent then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: It is not necessary to start the integral test at n=1. For example, to test

$$\sum_{n=4}^{\infty} \frac{1}{(n-2)^3} \text{ we use } \int_{4}^{\infty} \frac{1}{(x-2)^3} dx$$

Example: Determine whether the series $\sum_{n=1}^\infty \frac{\ln n}{n}$ diverges or converges.



Example: For what values of p is the series $\sum_{n=1}^\infty \frac{1}{n^p}$ convergent.

p-series

The p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent if p>1 and divergent if $p\leq 1.$

These series will be useful when we use the comparison tests.

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n then $\sum a_n$ is also convergent.

b) If $\sum b_n$ is convergent and $a_n \ge b_n$ for all n then $\sum a_n$ is also convergent.

Examples: Determine whether the following series converge or diverge:

1)
$$\sum_{n=1}^{\infty} \frac{3}{5n^3 + 2n + 1}$$

2)
$$\sum_{n=1}^{\infty} \frac{n+5}{n^{3/2} - 2\sqrt{n} - 1}$$

3)
$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

4)
$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 5}$$

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0\,,$ then either both series converge or both diverge.

Example: Determine whether the following series converge or diverge.

1)
$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 5}$$

2)
$$\sum_{n=1}^{\infty} \frac{2}{3^n - 1}$$

3)
$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

$$\text{4)} \sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$$