

Trigonometric Integrals

In this section we will investigate methods for integrating the product of various trigonometric functions like

$$\cos^2 x \sin x, \tan^4 x \sec^4 x, \dots$$

For many of these types of integrals we will need some identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$1 = \cos^2 x + \sin^2 x$$

$$\cot^2 x = \csc^2 x - 1$$

$$\tan^2 x = \sec^2 x - 1$$

There are a series of rules that depend on the trig functions in the integral and the parity of the powers. Because of the above identities we should try to pair trig functions together:

$$\sin x \quad \text{with} \quad \cos x$$

$$\tan x \quad \text{with} \quad \sec x$$

$$\csc x \quad \text{with} \quad \cot x$$

For sine and cosine

1) If the power of sine and cosine are even then use the half angle identities.

Example: Find

$$\int \sin^2 x \cos^2 x dx$$

2) If one of the powers is odd, factor out one trig function (for substitution) and use an identity on the remaining term.

Given

$$\int \sin^{2k+1} x \cos^n x dx \quad \text{where } k \text{ is a positive integer.}$$

or if

$$\int \cos^{2k+1} x \sin^n x dx \quad \text{where } k \text{ is a positive integer.}$$

Example: Find

1) $\int \sin^4 x \cos^3 x dx$

2) $\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$

For secant and tangent

3) If there is only $\tan x$ to an even positive power in the integral then use the identity $\tan^2 x = \sec^2 x - 1$ until there is only one $\tan^2 x$ remaining.

Example: Find

1) $\int \tan^4 x dx$

2) $\int \tan^6 x dx$

3) $\int \tan^8 x dx$

4) If secant has an even power then factor $\sec^2 x$ out of the term (for substitution) and convert the remaining to tangent using the identity.

Given $\int \sec^{2k} x \tan^n x dx$ where k is a positive integer.

Example find $\int \tan^3 x \sec^4 x dx$

5) If tangent has an odd power then we factor out $\tan x \sec x$ (for substitution) and convert the remaining tangents to secants.

Given

$$\int \tan^{2k+1} x \sec^n x dx \text{ where } n \text{ and } k \text{ are positive integers}$$

Example: Find $\int \tan^3 x \sec^3 x dx$

6) If tangent has an odd power by itself convert all but one tanx to secx.

Example: Find

$$\int \tan^5 x dx$$

Sometimes we must be clever.

Example: Find

$$\int \sec^3 x dx$$