Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S03003: Winter 2013

Name: Y. Lamontagne
Student ID: ____________

Test 1

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Evaluate the definite integral of $f(x) = -3x^2 + 4x + 3$ on $[-2, 1]$ using the definition of the definite integral.

$$\int_{-2}^{1} f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n} = \frac{1-(-2)}{n} = \frac{3}{n}$

$x_i = a + i\Delta x = -2 + \frac{3i}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[-3\left(-2+\frac{3i}{n}\right)^2 + 4\left(-2+\frac{3i}{n}\right) + 3\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[-3\left(4 - 12\frac{i}{n} + \frac{9i^2}{n^2}\right) - 8 + \frac{12i}{n} + 3\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[-12 + 36\frac{i}{n} - \frac{27i^2}{n^2} - 8 + \frac{12i}{n} + 3\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{-27i^2}{n^2} + 48\frac{i}{n} - 17\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\frac{-27}{n^2}\sum_{i=1}^{n} i^2 + \frac{48}{n}\sum_{i=1}^{n} i - \sum_{i=1}^{n} 17\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\frac{-\cancel{27}^{\,9}}{n^{\cancel{2}}} \frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,2} + \frac{\cancel{48}^{\,24}}{\cancel{n}} \frac{n(n+1)}{\cancel{2}} - 17n\right]$$

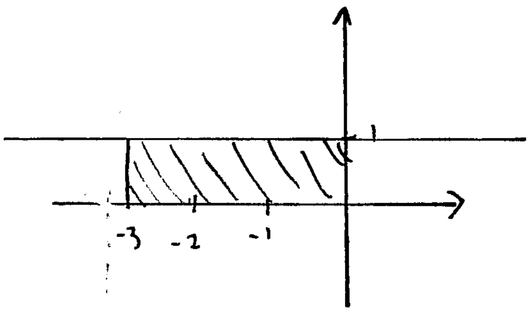
$$= \lim_{n\to\infty} \left[\frac{-27(n+1)}{n}\frac{(2n+1)}{2n} + \frac{72}{n}(n+1) - 51\frac{n}{n}\right]$$

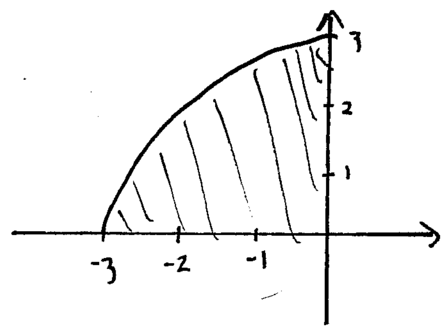
$$= \frac{-27}{1}\cdot\frac{2}{2} + 72\cdot\frac{1}{1} - 51$$

$$= -6$$

Question 2. *(5 marks)* Evaluate the integral by interpreting it in terms of areas.

$$\int_{-3}^{0} 1+\sqrt{9-x^2}\,dx = \int_{-3}^{0} 1\,dx + \int_{-3}^{0} \sqrt{9-x^2}\,dx$$





$$= 3(1) + \frac{\pi(3)^2}{4}$$

$$= 3 + \frac{9\pi}{4}$$

Question 3. *(5 marks)* If

$$\int_0^{\pi} f(x) = \sqrt{2}$$

and $f(x) = f(-x)$ for all $x \in \mathbb{R}$ then evaluate

$$\int_{-\pi}^{\pi} 2f(x) + \frac{x^2 \sin x}{1+x^6}\,dx = \int_{-\pi}^{\pi} 2f(x)\,dx + \int_{-\pi}^{\pi} \frac{x^2 \sin x}{1+x^6}\,dx$$

$$= 2\int_{-\pi}^{\pi} f(x)\,dx + 0$$ since $g(x)$ is odd

$$= 2\cdot 2\int_{0}^{\pi} f(x)\,dx$$ since $f(x)$ is even

$$= 4\sqrt{2}$$

Let $g(x) = \dfrac{x^2 \sin x}{1+x^6}$

$$g(-x) = \frac{(-x)^2 \sin(-x)}{1+(-x)^6}$$

$$= \frac{x^2(-\sin x)}{1+x^6}$$

$$= -g(x)$$

$\therefore$ $g(x)$ is an odd function

Question 4. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{\csc 3x \cot 3x}{1+\csc^2 3x}\,dx$$

$u = \csc 3x$

$du = -\csc 3x \cot 3x \cdot 3\,dx$

$\frac{-du}{3} = \csc 3x \cot 3x\,dx$

$= \int \frac{1}{1+u^2} \frac{-du}{3}$

$= \frac{-1}{3} \int \frac{1}{1+u^2}\,du$

$= \frac{-1}{3} \arctan u + C = \frac{-1}{3} \arctan(\csc 3x) + C.$

Question 5. Given

$$h(x) = \int_{2e^x}^{e^{2x}} u^u\,du = \int_{2e^x}^{1} u^u\,du + \int_{1}^{e^{2x}} u^u\,du = -\int_{1}^{2e^x} u^u\,du + \int_{1}^{e^{2x}} u^u\,du$$

a. *(2 marks)* Rewrite $h(x)$ as the sum of two integrals with a constant as the lower bound.

b. *(1 mark)* Rewrite the two integrals of part a. as composite functions with an integral as the outer function.

c. *(2 marks)* Using part b. and the 2[nd] FTC determine $h'(x)$.

b) $h(x) = -f(g_1(x)) + f(g_2(x))$ where $f(x) = \int_1^x u^u\,du$

$g_1(x) = 2e^x$

$g_2(x) = e^{2x}$

c) $h'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x)$

where $f'(x) = x^x$ by 2^{nd} FTC

$g_1'(x) = 2e^x$

$g_2'(x) = 2e^{2x}$

$= -(2e^x)^{2e^x} 2e^x + (e^{2x})^{e^{2x}} 2e^{2x}$

Question 6. Given

$$f(x) = \frac{1}{x}, \quad [1, e^2]$$

a. *(2 marks)* Find the average value of f on the given interval.

b. *(1 mark)* Find c such that $f_{ave} = f(c)$.

c. *(2 marks)* Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

a)

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{e^2-1}\int_1^{e^2} \frac{1}{x}\,dx$$

$$= \frac{1}{e^2-1}\left[\ln|x|\right]_1^{e^2}$$

$$= \frac{1}{e^2-1}\left[\ln e^2 - \ln 1\right]$$

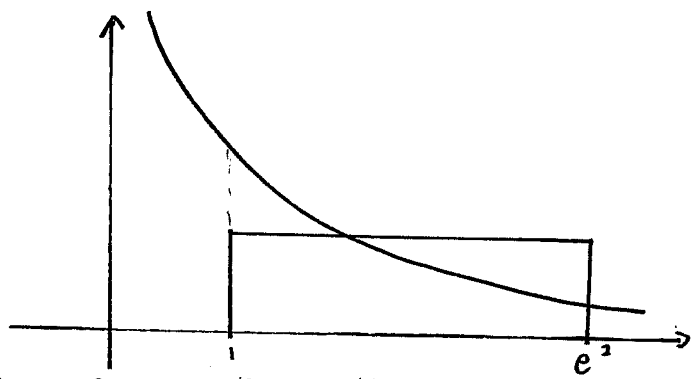
$$= \frac{2}{e^2-1}$$

b) $f_{ave} = f(c)$

$$\frac{2}{e^2-1} = \frac{1}{c}$$

$$c = \frac{e^2-1}{2}$$

c)



Question 7. *(5 marks)* Estimate the area under the graph of $f(x) = \tan x$ from $x = -\pi/3$ to $x = \pi/6$ using three rectangles and using the left endpoints. Sketch the curve and the approximating rectangles.

$$\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{6} - \left(-\frac{\pi}{3}\right)}{3} = \frac{\frac{3\pi}{6}}{3} = \frac{\pi}{6}$$

$$x_i = a + i\Delta x = -\frac{\pi}{3} + i\frac{\pi}{6}$$

$x_0 = -\pi/3$

$x_1 = -\pi/6$

$x_2 = 0$

$$\text{Area} \approx \sum_{i=0}^{2} |f(x_i)\Delta x|$$

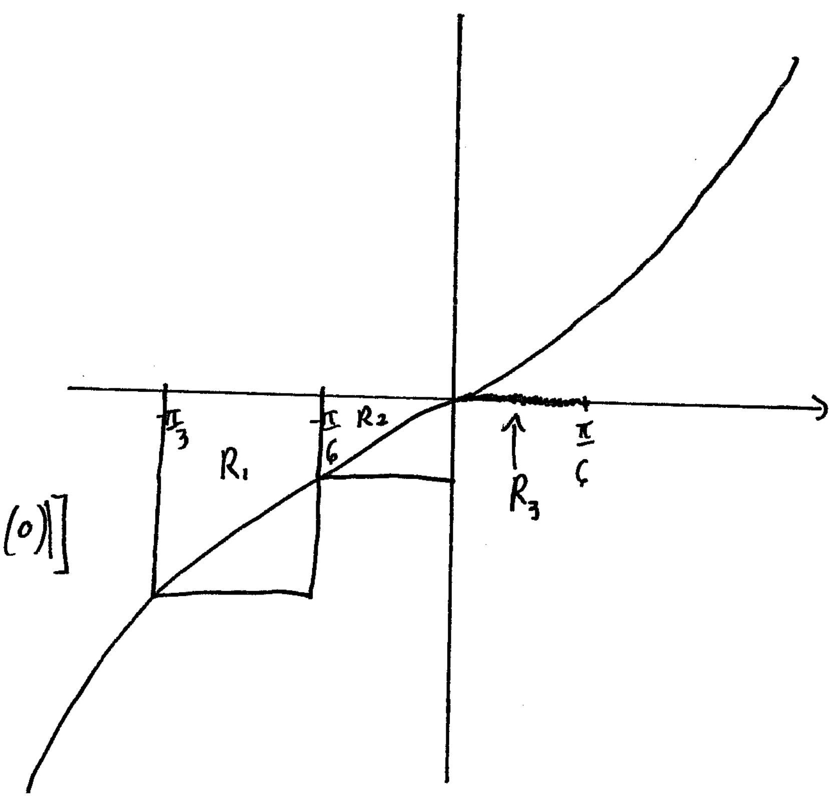
$$= \Delta x\left[|f(x_0)| + |f(x_1)| + |f(x_2)|\right]$$

$$= \frac{\pi}{6}\left[\left|\tan\left(-\frac{\pi}{3}\right)\right| + \left|\tan\left(-\frac{\pi}{6}\right)\right| + \left|\tan(0)\right|\right]$$

$$= \frac{\pi}{6}\left[|-\sqrt{3}| + \left|\left(-\frac{1}{\sqrt{3}}\right)\right|\right]$$

$$= \frac{\pi}{6}\left[\frac{+3+1}{\sqrt{3}}\right]$$

$$= +\frac{2\pi}{3\sqrt{3}}$$



Question 8. *(5 marks)* Prove: If a and b are positive numbers then

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$$

$$LHS = \int_0^1 x^a(1-x)^b dx = \int_1^0 (1-u)^a u^b (-du)$$

$u = 1-x$ *

$du = -dx$

$-du = dx$

$u(1) = 1-1 = 0$

$u(0) = 1-0 = 1$

* $x = 1-u$

$$= -\int_1^0 u^b(1-u)^a du$$

$$= \int_0^1 u^b(1-u)^a du = RHS$$

Question 9. *(5 marks)* Find a function f such that

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

and the line $y = x$ is tangent to the graph f.

at what x does f(x) have slope equal to 1 i.e. $y = 1\cdot x$

$1 = f'(x)$

$1 = \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2} = 1$

$1-x^2 = 1$

$0 = x^2$

$0 = x$

$\therefore$ $f(0)=0$ is an initial point since $y=x$ is tangent to $f(x)$

$f(x) = \int f'(x)\,dx$

$= \int \frac{1}{\sqrt{1-x^2}} dx$

$= \arcsin x + C$

$0 = f(0)$

$0 = \arcsin 0 + C$

$0 = 0 + C$

$0 = C$

$\therefore f(x) = \arcsin x.$

Bonus Question. *(3 marks)*
The error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

a. *(1 mark)* Show that

$$\int_a^b e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}[\operatorname{erf}(b) - \operatorname{erf}(a)].$$

b. *(2 marks)* Show that the function

$$y = e^{x^2} \operatorname{erf}(x)$$

satisfies the differential equation

$$y' = 2xy + \frac{2}{\sqrt{\pi}}$$

a)

$$\operatorname{erf}(b) - \operatorname{erf}(a) = \frac{2}{\sqrt{\pi}} \int_0^b e^{-t^2} dt - \frac{2}{\sqrt{\pi}} \int_0^a e^{-t^2} dt$$

$$\operatorname{erf}(b) - \operatorname{erf}(a) = \frac{2}{\sqrt{\pi}} \left[\int_a^0 e^{-t^2} dt + \int_0^b e^{-t^2} dt \right]$$

$$\frac{\sqrt{\pi}}{2} [\operatorname{erf}(b) - \operatorname{erf}(a)] = \int_a^b e^{-t^2} dt$$

b)

$$y' = e^{x^2}(2x) \operatorname{erf}(x) + e^{x^2} (\operatorname{erf}(x))'$$

$$= 2x \underbrace{e^{x^2} \operatorname{erf}(x)}_{y} + e^{x^2} \frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$= 2xy + \frac{2}{\sqrt{\pi}}$$