

## Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. (5 marks) Evaluate the definite integral

$$\int_0^{\sqrt[3]{\pi/4}} x^2 \tan^2 x^3 dx = \int_0^{\pi/4} \tan^2 u \frac{du}{3} = \frac{1}{3} \int_0^{\pi/4} \sec^2 u - 1 du$$

$$u = x^3$$

$$du = 3x^2 dx$$

$$\frac{du}{3} = x^2 dx$$

$$u(\sqrt[3]{\pi/4}) = \pi/4$$

$$u(0) = 0$$

$$= \frac{1}{3} \left[ \tan u - u \right]_0^{\pi/4}$$

$$= \frac{1}{3} \left[ \tan \frac{\pi}{4} - \frac{\pi}{4} \right]$$

$$= \frac{1}{3} \left[ 1 - \frac{\pi}{4} \right]$$

Question 2. (5 marks) Evaluate the indefinite integral

$$\int \frac{x^3 + x^2 + 1}{x^3 + x^2 - 6x} dx = \int \frac{x^3 + x^2 - 6x + 6x + 1}{x^3 + x^2 - 6x} dx$$

$$= \int \frac{x^3 + x^2 - 6x}{x^3 + x^2 - 6x} + \frac{6x + 1}{x^3 + x^2 - 6x} dx$$

$$= \int 1 + \frac{6x + 1}{x^3 + x^2 - 6x} dx$$

$$= x + \int \frac{6x + 1}{x(x^2 + x - 6)} dx$$

$$= x + \int \frac{6x + 1}{x(x+3)(x-2)} dx$$

$$= x + \int \frac{-\frac{1}{6}}{x} - \frac{17/15}{x+3} + \frac{13/10}{x-2} dx$$

$$= x - \frac{1}{6} \ln|x| - \frac{17}{15} \ln|x+3|$$

$$+ \frac{13}{10} \ln|x-2| + C$$

$$\frac{6x+1}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}$$

$$6x+1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3)$$

Let  $x=0$

$$1 = -6A \Rightarrow A = -\frac{1}{6}$$

Let  $x=2$

$$13 = C(2)(5) \Rightarrow C = \frac{13}{10}$$

Let  $x=-3$

$$-17 = -15B \Rightarrow B = \frac{-17}{15}$$

Question 3. (5 marks) Find the exact length of the curve

$$y = \ln(\cos x), \quad 0 \leq x \leq \frac{\pi}{3}$$

$$y' = \frac{1}{\cos x} \cdot -\sin x = -\tan x$$

$$S = \int_0^{\pi/3} \sqrt{1 + (y')^2} dx$$

$$= \int_0^{\pi/3} \sqrt{1 + (-\tan x)^2} dx$$

$$= \int_0^{\pi/3} \sqrt{1 + \tan^2 x} dx$$

$$= \int_0^{\pi/3} \sqrt{\sec^2 x} dx$$

$$= \int_0^{\pi/3} |\sec x| dx$$

$$= \int_0^{\pi/3} \sec x dx$$

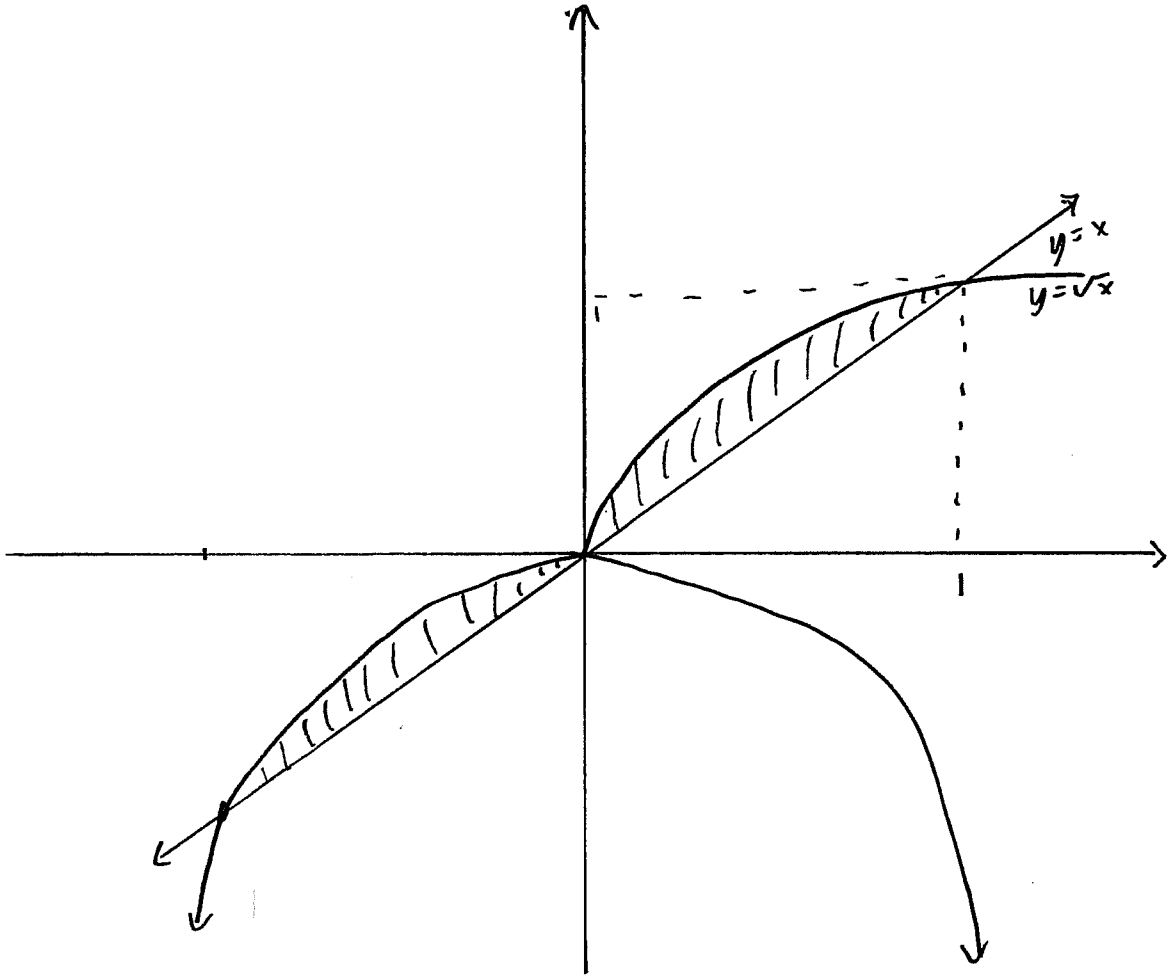
$$= \left[ \ln |\sec x + \tan x| \right]_0^{\pi/3}$$

$$= \ln |\sec \pi/3 + \tan \pi/3| - \ln |\sec 0 + \tan 0|$$

$$= \ln |2 + \sqrt{3}|$$

**Question 4.** (5 marks) Sketch the region(s) enclosed by the given curves and set up the integral to find the total area of the region(s).

$$y = x, y = \sqrt{x}, y = -x^2$$



$$A = \int_{-1}^0 -x^2 - x \, dx + \int_0^1 \sqrt{x} - x \, dx$$

Question 5. (5 marks) Determine whether the integral is convergent or divergent. Evaluate the integral if convergent.

$$\begin{aligned} \int_{2\sqrt{3}}^{\infty} \frac{1}{x\sqrt{x^2-3}} dx &= \lim_{b \rightarrow \infty} \int_{2\sqrt{3}}^b \frac{1}{x\sqrt{x^2-3}} dx \\ &= \lim_{b \rightarrow \infty} \left[ \frac{1}{\sqrt{3}} \operatorname{arcsec} \frac{x}{\sqrt{3}} \right]_{2\sqrt{3}}^b \\ &= \lim_{b \rightarrow \infty} \left[ \frac{1}{\sqrt{3}} \operatorname{arcsec} \frac{b}{\sqrt{3}} \xrightarrow{\frac{\pi}{2}} \frac{1}{\sqrt{3}} \operatorname{arcsec} \frac{2\sqrt{3}}{\sqrt{3}} \right] \\ &= \frac{\pi}{2\sqrt{3}} - \frac{\pi}{3\sqrt{3}} \end{aligned}$$

Question 6. (5 marks) Evaluate the definite integral

$$\begin{aligned}
 \int_3^5 \sqrt{x^2 - 2x - 3} \, dx &= \int_3^5 \sqrt{(x-1)^2 - 4} \, dx && x-1 = 2\sec\theta \\
 &&& dx = 2\sec\theta \tan\theta \\
 x^2 - 2x - 3 &= x^2 - 2x + 1 - 1 - 3 && 5-1 = 2\sec\theta \\
 &= (x-1)^2 - 4 && 2 = \sec\theta \Rightarrow \theta = \frac{\pi}{3} \\
 &&& 3-1 = 2\sec\theta \\
 &&& 1 = \sec\theta \Rightarrow \theta = 0 \\
 &= 2 \int_0^{\pi/3} \sqrt{4\sec^2\theta - 4} \sec\theta \tan\theta \, d\theta \\
 &= 2 \int_0^{\pi/3} \sqrt{4(\sec^2\theta - 1)} \sec\theta \tan\theta \, d\theta \\
 &= 2 \int_0^{\pi/3} \sqrt{4\tan^2\theta} \sec\theta \tan\theta \, d\theta \\
 &= 4 \int_0^{\pi/3} \sec\theta \tan^2\theta \, d\theta \\
 &= 4 \int_0^{\pi/3} \sec\theta (\sec^2\theta - 1) \, d\theta \\
 &= 4 \int_0^{\pi/3} \sec^3\theta \, d\theta - 4 \int_0^{\pi/3} \sec\theta \, d\theta \\
 &= 4 \left[ \sqrt{3} + \frac{1}{2} \ln(2 + \sqrt{3}) \right] - 4 \left[ \ln|\sec\theta + \tan\theta| \right]_0^{\pi/3} \\
 &= 4\sqrt{3} + 2 \ln(2 + \sqrt{3}) - 4 \left[ \ln|\sec \frac{\pi}{3} + \tan \frac{\pi}{3}| \right. \\
 &\quad \left. - \ln|\sec 0 + \tan 0| \right] \\
 &= 4\sqrt{3} + 2 \ln(2 + \sqrt{3}) - 4 \ln(2 + \sqrt{3}) \\
 &= 4\sqrt{3} - 2 \ln(2 + \sqrt{3})
 \end{aligned}$$

Question 7. (5 marks) Evaluate the indefinite integral

$$\int \arctan(\sqrt{x}) dx = uv - \int v du$$

$$u = \arctan \sqrt{x}$$

$$du = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} dx$$

$$v = x$$

$$dv = dx$$

$$= x \arctan \sqrt{x} - \int x \cdot \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} dx$$

$$u = \sqrt{x}$$

$$du = \frac{1}{2\sqrt{x}} dx$$

$$= x \arctan \sqrt{x} - \int u^2 \frac{1}{1+u^2} du$$

$$= x \arctan \sqrt{x} - \int \frac{u^2+1-1}{1+u^2} du$$

$$= x \arctan \sqrt{x} - \int \frac{\cancel{1+u^2}}{1+u^2} du + \int \frac{1}{u^2+1} du$$

$$= x \arctan \sqrt{x} - \int 1 du + \arctan u + C$$

$$= x \arctan \sqrt{x} - u + \arctan \sqrt{x} + C$$

$$= x \arctan \sqrt{x} - \sqrt{x} + \arctan \sqrt{x} + C$$

**Question 8.** (5 marks) Find the value(s) of the constant  $p$  for which the integral

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x^p} dx$$

converges. Evaluate the integral for this value(s) of  $p$ .

if  $p=1$   $\lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \left[ \ln|x| \right]_a^1 = \lim_{a \rightarrow 0^+} \left[ \ln 1 - \ln a \right] = \lim_{a \rightarrow 0^+} \left[ 0 - \ln a \right] = \infty$   $\therefore$  if  $p=1$  the integral diverges to  $\infty$

if  $p \neq 1$   $\lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \int_a^1 x^{-p} dx = \lim_{a \rightarrow 0^+} \left[ \frac{x^{-p+1}}{-p+1} \right]_a^1 = \lim_{a \rightarrow 0^+} \left[ \frac{1^{-p+1}}{-p+1} - \frac{a^{-p+1}}{-p+1} \right]$

if  $p > 1$  then  $\lim_{a \rightarrow 0^+} \frac{a^{-p+1}}{-p+1} = -\infty$  so the integral diverges to  $\infty$ .

if  $p < 1$  then  $\lim_{a \rightarrow 0^+} \frac{a^{-p+1}}{-p+1} = 0$  so the integral converges to  $\frac{1}{1-p}$



Question 9. (5 marks) Evaluate the definite integral

$$\int_0^{\frac{\pi}{4}} e^{\cos x} \sin 2x \, dx = \int_0^{\frac{\pi}{4}} 2 \sin x \cos x e^{\cos x} \, dx$$

$z = \cos x \quad dz = -\sin x \, dx$   
 $z(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \quad -dz = \sin x \, dx$   
 $z(0) = \cos 0 = 1$

$$= -2 \int_1^{\frac{1}{\sqrt{2}}} z e^z \, dz$$

$u = z$   
 $v = e^z$   
 $du = dz$   
 $dv = e^z \, dz$

$$= -2 \left[ [uv]_1^{\frac{1}{\sqrt{2}}} - \int_1^{\frac{1}{\sqrt{2}}} u \, du \right]$$
$$= -2 \left[ [z e^z]_1^{\frac{1}{\sqrt{2}}} - \int_1^{\frac{1}{\sqrt{2}}} e^z \, dz \right]$$
$$= -2 \left[ \frac{1}{\sqrt{2}} e^{\frac{1}{\sqrt{2}}} - e - [e^z]_1^{\frac{1}{\sqrt{2}}} \right] = -2 \left[ \frac{e^{\frac{1}{\sqrt{2}}}}{\sqrt{2}} - e - e^{\frac{1}{\sqrt{2}}} + e \right]$$
$$= 2e^{\frac{1}{\sqrt{2}}} - \frac{2}{\sqrt{2}} e^{\frac{1}{\sqrt{2}}}$$

**Bonus Question.** (3 marks) Find the exact length of the curve

$$y = \int_2^{\frac{e^x+1}{e^x-1}} \frac{1}{z} dz \quad [\ln 2, \ln 3]$$

$$S = \int_{\ln 2}^{\ln 3} \sqrt{1 + (y')^2} dx$$

$$y' = \frac{1}{\frac{e^x+1}{e^x-1}} \cdot \frac{e^x(e^x-1) - (e^x+1)e^x}{(e^x-1)^2}$$

$$= \frac{-2e^x}{e^{2x}-1}$$

$$S = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{-2e^x}{e^{2x}-1}\right)^2} dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4e^{2x}}{(e^{2x}-1)^2}} dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{(e^{2x}-1)^2 + 4e^{2x}}{(e^{2x}-1)^2}} dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{(e^{2x}-1)^2}} dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\sqrt{e^{4x} + 2e^{2x} + 1}}{(e^{2x}-1)} dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\sqrt{(e^{2x}+1)^2}}{(e^{2x}-1)} dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x}+1}{e^{2x}-1} dx$$

add and remove

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x} + e^{2x} - e^{2x} + 1}{e^{2x}-1} dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{-e^{2x} + 1 + 2e^{2x}}{e^{2x}-1} dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{-e^{2x} + 1}{e^{2x}-1} dx + \int_{\ln 2}^{\ln 3} \frac{2e^{2x}}{e^{2x}-1} dx$$

$$\begin{aligned}
&= - \int_{\ln 2}^{\ln 3} \frac{e^{2x} - 1}{e^{2x} - 1} dx + \int_{\ln 2}^{\ln 3} \frac{2e^{2x}}{e^{2x} - 1} dx \\
&= - [x]_{\ln 2}^{\ln 3} + [\ln |e^{2x} - 1|]_{\ln 2}^{\ln 3} \\
&= -\ln 3 + \ln 2 + [\ln |e^{2\ln 3} - 1| - \ln |e^{2\ln 2} - 1|] \\
&= \ln \frac{2}{3} + \ln |e^{\ln 3^2} - 1| - \ln |e^{\ln 2^2} - 1| \\
&= \ln \frac{2}{3} + \ln |9 - 1| - \ln |4 - 1| \\
&= \ln \frac{2}{3} + \ln 8 - \ln 3 \\
&= \ln \frac{2}{3} + \ln \frac{8}{3} \\
&= \ln \frac{2 \cdot 8}{3 \cdot 3} \\
&= \ln \frac{16}{9}
\end{aligned}$$