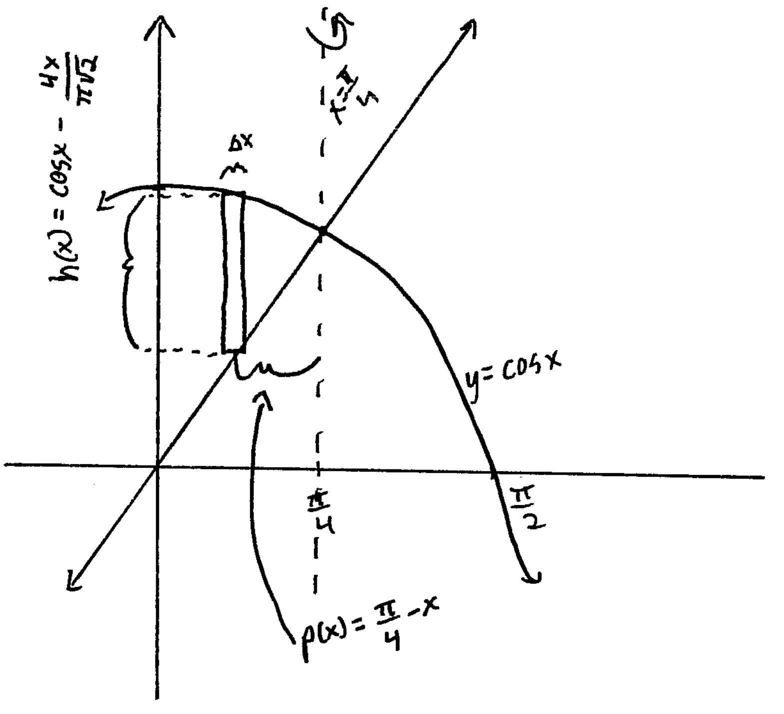
Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S03003: Winter 2013

Name: Y. Lamontagne
Student ID: ________________

Test 3

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

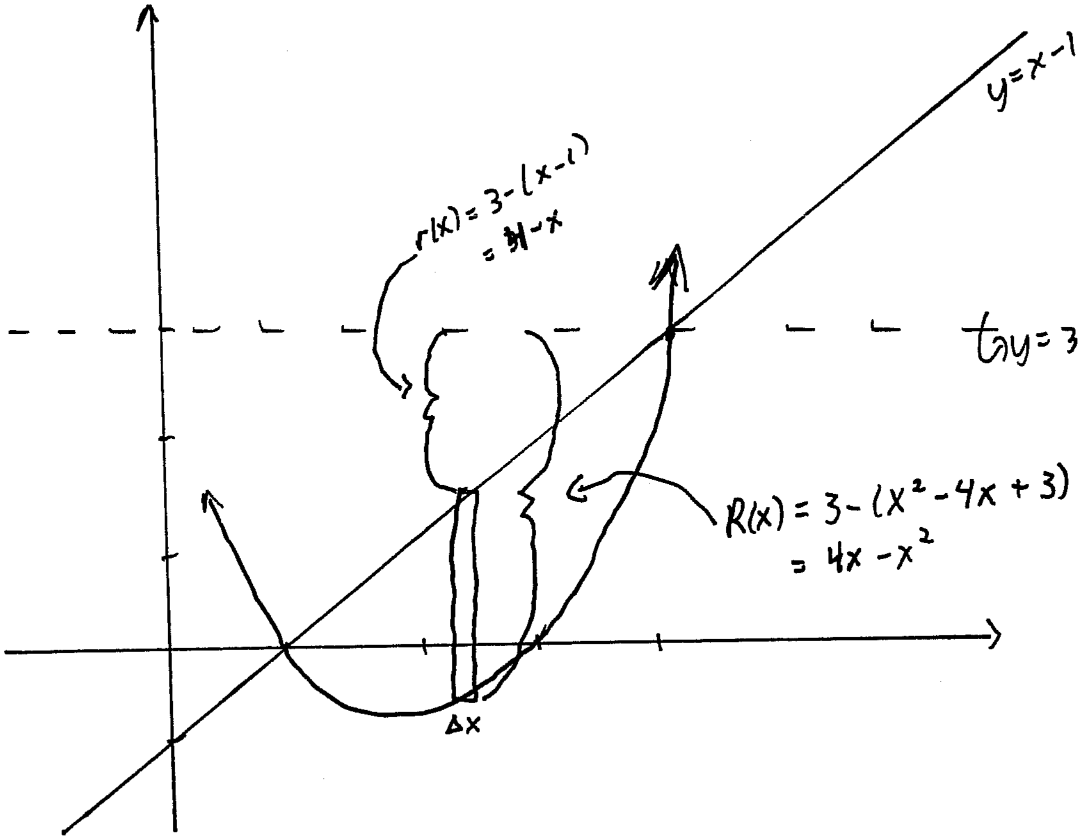
Question 1. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region in the first quadrant bounded by the graphs of $y = \cos x$, $y = \frac{4x}{\pi\sqrt{2}}$ and $x = 0$ rotated about the line $x = \frac{\pi}{4}$.



$$\Delta V = 2\pi\, p(x)\, h(x)\, \Delta x$$
$$= 2\pi \left(\frac{\pi}{4} - x\right)\left[\cos x - \frac{4x}{\pi\sqrt{2}}\right]\Delta x$$

$$V = \int_0^{\frac{\pi}{4}} 2\pi\left(\frac{\pi}{4} - x\right)\left[\cos x - \frac{4x}{\pi\sqrt{2}}\right]dx$$

Question 2. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region in the first quadrant bounded by the graphs of $x-y=1$, $y=x^2-4x+3$ and $x=0$ rotated about the line $y=3$.



For $y=x^2-4x+3$

x-int:

$0=x^2-4x+3$

$0=(x-3)(x-1)$

$x=3 \qquad x=1$

intersection of two curve: $x-1=x^2-4x+3$

$0=x^2-5x+4$

$0=(x-1)(x-4)$

$x=1 \qquad x=4$

$$\Delta V=\pi\left[(R(x))^2-(r(x))^2\right]\Delta x=\pi\left[(4x-x^2)^2-(4-x)^2\right]\Delta x$$

$$V=\int_1^4 \pi\left[(4x-x^2)^2-(4-x)^2\right]dx$$

Question 3. *(5 marks)*

a. *(2 marks)* Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$\left\{\tan(1), 2\tan\left(\frac{1}{2}\right), 3\tan\left(\frac{1}{3}\right), 4\tan\left(\frac{1}{4}\right), 5\tan\left(\frac{1}{5}\right), \ldots\right\}_{n=1}^{\infty}$$

b. *(3 marks)* Determine the limit of a_n as $n \to \infty$.

a) $a_n = n\tan\left(\frac{1}{n}\right)$

b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\tan\left(\frac{1}{n}\right)$ I.f. $\infty \cdot 0$

$= \lim_{n\to\infty} \dfrac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}$ I.f. $\frac{0}{0}$

$\lim_{x\to\infty} \dfrac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$ I.f. $\frac{0}{0}$

$\stackrel{H}{=} \lim_{x\to\infty} \dfrac{\sec^2\left(\frac{1}{x}\right) \cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}}$

$= \lim_{x\to\infty} \sec^2\left(\frac{1}{x}\right)$

$= \sec^2(0)$

$= 1$

Question 4. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=2}^{\infty}\left[e^{1/n}-e^{1/(n+1)}\right]$$

Lets look at the partial sum S_n

$$S_n = a_2 + a_3 + a_4 + \dots + a_{n-2} + a_{n-1} + a_n$$

$$= \left[e^{1/2}-\cancel{e^{\frac{1}{3}}}\right] + \left[\cancel{e^{1/3}}-\cancel{e^{\frac{1}{4}}}\right] + \left[\cancel{e^{1/4}}-\cancel{e^{1/5}}\right]$$

$$+ \dots + \left[\cancel{e^{\frac{1}{n-2}}}-\cancel{e^{\frac{1}{n-1}}}\right] + \left[\cancel{e^{\frac{1}{n-1}}}-\cancel{e^{\frac{1}{n}}}\right] + \left[\cancel{e^{1/n}}-e^{\frac{1}{n+1}}\right]$$

$$= e^{1/2} - e^{\frac{1}{n+1}}$$

$$\sum_{n=2}^{\infty}\left[e^{1/n}-e^{1/n+1}\right] = S = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left[e^{1/2}-e^{\cancel{\frac{1}{n+1}} \to 0}\right]$$

$$= e^{1/2} - e^{0} = e^{1/2} - 1$$

$\therefore$ the series is convergent.

Question 5. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=2}^{\infty} \frac{2^n - 3^{n-1}}{4^{n+1}}$$

Lets look at both series independently

$$\sum_{n=2}^{\infty} \frac{2^n}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{2^n}{4 4^n} = \sum_{n=2}^{\infty} \frac{1}{4}\left(\frac{2}{4}\right)^n = \sum_{n=2}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^n$$

conv. since geometric series where $r = \frac{1}{2} < 1$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^{n+2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{16}\left(\frac{1}{2}\right)^n = \frac{a}{1-r}$$

$$= \frac{1/16}{1-1/2} = 1/8$$

$$\sum_{n=2}^{\infty} \frac{3^{n-1}}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{3^{-1}3^n}{4 4^n} = \sum_{n=2}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^n$$

converges since geometric series where $r = \frac{3}{4} < 1$

$$= \sum_{n=0}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^{n+2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^2\left(\frac{3}{4}\right)^n = \frac{a}{1-r}$$

$$= \frac{\frac{1}{12}\left(\frac{3}{4}\right)^2}{1-\frac{3}{4}}$$

$$= \frac{\frac{1}{\cancel{12}}\left(\frac{\cancel{9}^{\,3}}{16}\right)}{\frac{1}{4}} = \frac{3}{16}$$

$$\sum_{n=2}^{\infty} \frac{2^n - 3^{n-1}}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{2^n}{4^{n+1}} - \sum_{n=2}^{\infty} \frac{3^{n-1}}{4^{n+1}}$$

$$= \frac{1}{8} - \frac{3}{16} = \frac{-1}{16}$$

Question 6. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=3}^{\infty} \frac{3n^2}{n(n+3)} = \sum_{n=3}^{\infty} \frac{3n^2}{n^2+3n}$$

$$\lim_{n\to\infty} \frac{3n^2}{n^2+3n} = 3 \neq 0$$

$\therefore$ diverges by n^{th} term divergence test.

Question 7. *(5 marks)* Determine whether the series is convergent or divergent.

$\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$

Let $a_n = \frac{1+\sin n}{10^n}$

$$a_n = \frac{1+\sin n}{10^n} \leq \frac{1+1}{10^n} = \frac{2}{10^n} = 2\left(\frac{1}{10}\right)^n = b_n$$

$\sum_{n=0}^{\infty} b_n$ is a convergent series since it is geometric and $r = \frac{1}{10} < 1$

$\therefore$ by Comparison test $\sum_{n=0}^{\infty} a_n$ is convergent.

Question 8. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$\sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln n}$ Lets determine whether the series is absolutely convergent

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=3}^{\infty} \frac{1}{n \ln n}$$

Let $f(x) = \frac{1}{x \ln x}$

- $f(x)$ is continuous for $x > 3$.
- $f(x)$ is positive for $x > 3$.
- $f'(x) = \frac{-1}{(x \ln x)^2} [\ln x + x \frac{1}{x}] < 0$

for $x > 3$ $\therefore$ $f(x)$ is decreasing

$$\int_3^{\infty} \frac{1}{x \ln x} dx$$

$$= \lim_{b \to \infty} \int_3^b \frac{1}{x \ln x} dx = \lim_{b \to \infty} \int_{\ln 3}^{\ln b} \frac{1}{u} du$$

$u = \ln x$ $\quad u(b) = \ln b$

$du = \frac{1}{x} dx$ $\quad u(3) = \ln 3$

$$= \lim_{b \to \infty} \left[\ln |u| \right]_{\ln 3}^{\ln b} = \lim_{b \to \infty} \left[\ln \ln b - \ln \ln 3 \right]$$

∞

$\therefore \sum_{n=3}^{\infty} \frac{1}{n \ln n}$ is divergent by the integral test.

$\therefore \sum \frac{(-1)^n}{n \ln n}$ is not absolutely convergent.

Lets determine whether the series is convergent

$\sum_{n=3}^{\infty} (-1)^n b_n$ where $b_n = \frac{1}{n \ln n}$, lets apply the alternating series test

① Let $f(x) = \frac{1}{x \ln x}$, $f'(x) < 0$ $\therefore$ $f(n+1) < f(n)$

$b_{n+1} < b_n$

② $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n \ln n} = 0$

$\therefore$ the series converges by the alternating series test

$\therefore$ the series is conditionaly convergent.

Question 9. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=4}^{\infty} \frac{(-1)^n 2^{n^2}}{n!}$$

Lets apply the ratio test : $a_n = \frac{(-1)^n 2^{n^2}}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} 2^{(n+1)^2}}{(n+1)!}}{\frac{(-1)^n 2^{n^2}}{n!}} \right|$$

$$= \lim_{n\to\infty} \frac{2^{n^2+2n+1} \cancel{n!}}{2^{n^2} \cancel{n!} (n+1)}$$

$$= \lim_{n\to\infty} \frac{2^{2n+1}}{n+1}$$

$$= \lim_{x\to\infty} \frac{2^{2x+1}}{x+1} \quad \text{I.F. } \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{2^{2x+1} \ln 2 \,(2)}{1} \overset{\infty}{} > 1$$

$\therefore$ diverges by ratio test.

Bonus Question. *(3 marks)* Let $\{b_n\}$ be a sequence of positive numbers that converge to $\frac{1}{2}$. Determine whether the given series is absolutely convergent

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 b_3 \ldots b_n}$$

Lets try to apply the ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1} b_1 b_2 b_3 \cdots b_n b_{n+1}}}{\frac{(-1)^n n!}{n^n b_1 b_2 b_3 \cdots b_n}} \right|$$

$$= \lim_{n\to\infty} \frac{n^n \cancel{b_1}\cancel{b_2}\cancel{b_3}\cdots \cancel{b_n}(n+1)!}{(n+1)^{n+1}\cancel{b_1}\cancel{b_2}\cancel{b_3}\cdots \cancel{b_n} b_{n+1} n!}$$

$$= \lim_{n\to\infty} \frac{n^n \cancel{n!}\cancel{(n+1)}}{(n+1)^n \cancel{(n+1)} b_{n+1} \cancel{n!}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \lim_{n\to\infty} \frac{1}{b_{n+1}}$$

$$= \frac{1}{e} \cdot \frac{1}{\lim\limits_{n\to\infty} b_{n+1}}$$

$$= \frac{1}{e} \cdot \frac{1}{\frac{1}{2}}$$

$$= \frac{2}{e} < 1$$

$\therefore$ absolutely convergent by ratio test.

$$y = \lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x \quad \text{i.e. } 1^\infty$$

$$\ln y = \ln \lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$$

$$\ln y = \lim_{x\to\infty} \ln \left(\frac{x}{x+1}\right)^x$$

$$\ln y = \lim_{x\to\infty} x \ln\left(\frac{x}{x+1}\right)$$

$$\ln y = \lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}}$$

$$\ln y = \lim_{x\to\infty} \frac{\frac{1}{\frac{x}{x+1}} \cdot \frac{x+1-x}{(x+1)^2}}{\frac{-1}{x^2}}$$

$$\ln y = \lim_{x\to\infty} \frac{-x^2}{x^2+x}$$

$$\ln y = -1$$

$$y = e^{-1}$$