**Dawson College: Calculus II** (SCIENCE): 201-NYB-05-S09: Winter 2013

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# Test 1

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

**Formulae:**

$\sum_{i=1}^{n} c = cn$ where $c$ is a constant $\qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

**Question 1.** *(5 marks)* Evaluate the definite integral of $f(x) = -3x^2+1$ on $[-1,3]$ using the definition of the definite integral.

$$\int_{-1}^{3} -3x^2+1\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \Delta x = \frac{b-a}{n} = \frac{3-(-1)}{n} = \frac{4}{n}$$

$$x_i = a + i\Delta x = -1 + \frac{4i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[-3\left(-1+\frac{4i}{n}\right)^2 + 1\right]\frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[-3 + 24\frac{i}{n} - \frac{48i^2}{n^2} + 1\right]$$

$$= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^{n} \left[-2 + 24\frac{i}{n} - \frac{48i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{4}{n} \left[\sum_{i=1}^{n} -2 + \frac{24}{n}\sum_{i=1}^{n} i - \frac{48}{n^2}\sum_{i=1}^{n} i^2\right]$$

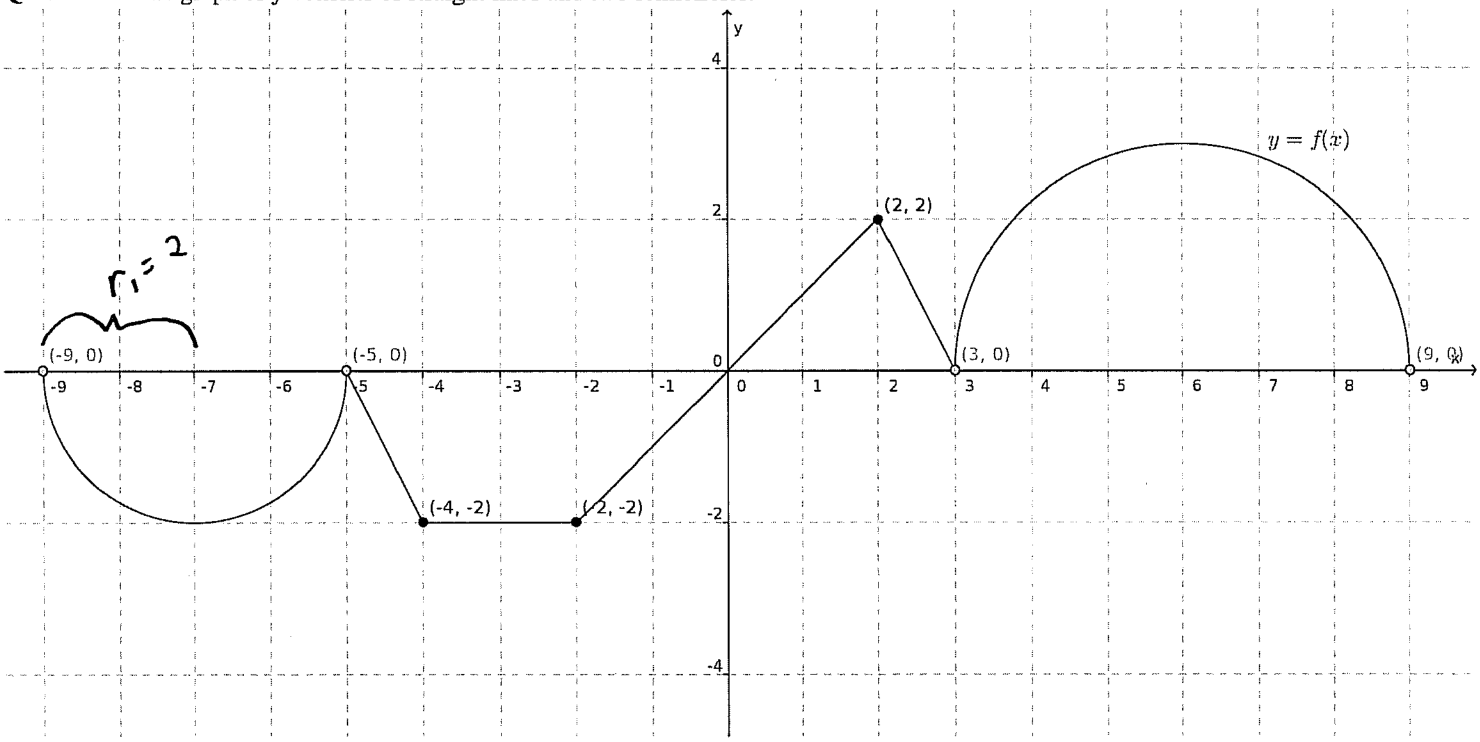
$$= \lim_{n\to\infty} \frac{4}{n} \left[-2n + \frac{\cancel{24}^{12}}{\cancel{n}} \frac{\cancel{n}(n+1)}{\cancel{2}} - \frac{48}{n^2} \frac{\cancel{n}(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[-8 + 48\frac{(n+1)}{n} - 32\frac{(n+1)}{n}\cdot\frac{(2n+1)}{n}\right]$$

$$= -8 + 48 - 32\cdot 2$$

$$= -24$$

**Question 2.** The graph of $f$ consists of straight lines and two semicircles.




Use the graph of the find the exact value of the following integrals.

a. *(1 mark)*

$$\int_{-9}^{-7} f(x)\,dx = \frac{\pi(2)^2}{4} = -\pi$$

b. *(1 mark)*

$$\int_{-1}^{1} f(x)\,dx = 0$$

c. *(1 mark)*

$$\int_{-3}^{0} f(x)\,dx = (-2)1 + \frac{2(-2)}{2} = -4$$

d. *(1 mark)*

$$\int_{0}^{3} f(x)\,dx = \frac{3(2)}{2} = 3$$

e. *(1 mark)*

$$\int_{-3}^{3} f(x)\,dx = -1$$

**Question 3.** *(5 marks)* If

$$\int_3^5 (f(x)-2)\,dx = 7, \quad \int_3^0 2f(x)\,dx = 3,$$

and $f(x) = f(-x)$ for all $x \in \mathbb{R}$ find

$$\int_{-5}^5 f(x)\,dx. = 2\int_0^5 f(x)\,dx \quad \text{since } f(x) \text{ is even}$$

$$= 2\left[\int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx\right]$$

$$= 2\left[\frac{-3}{2} + 11\right]$$

$$= 19$$

① $\int_3^0 2f(x)\,dx = 3$

$\int_0^3 2f(x)\,dx = -3$

$\int_0^3 f(x)\,dx = \frac{-3}{2}$

② $\int_3^5 (f(x)-2)\,dx = 7$

$\int_3^5 f(x)\,dx - \int_3^5 2\,dx = 7$

$\int_3^5 f(x)\,dx = 7 + \int_3^5 2\,dx$

$= 7 + [2x]_3^5$

$= 7 + [10-6]$

$= 11$

**Question 4.** *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{x^5}{\sqrt{x^3+\pi}}\,dx = \int \frac{x^3x^2}{\sqrt{x^3+\pi}}\,dx$$

$u = x^3+\pi$

$du = 3x^2dx$

$\frac{du}{3} = x^2dx$

$u-\pi = x^3$

$$= \int \frac{(u-\pi)}{\sqrt{u}}\,\frac{du}{3}$$

$$= \frac{1}{3}\int (u-\pi)\,u^{-1/2}\,du$$

$$= \frac{1}{3}\int u^{1/2} - \pi u^{-1/2}\,du$$

$$= \frac{1}{3}\left[\frac{2u^{3/2}}{3} - 2\pi u^{1/2}\right] + C$$

$$= \frac{2}{9}(x^3+\pi)^{3/2} - \frac{2\pi}{3}\sqrt{x^3+\pi} + C$$

**Question 5.** Given

$$h(x)=\int_{\arctan 3x}^{\cot 2x} u\sqrt[3]{\sin u}\,du \;=\; \int_{\arctan 3x}^{0} u\sqrt[3]{\sin u}\,du + \int_{0}^{\cot 2x} u\sqrt[3]{\sin u}\,du$$

a. *(2 marks)* Rewrite $h(x)$ as the sum of two integrals with a constant as the lower bound.

b. *(1 mark)* Rewrite the two integrals of part a. as composite functions with an integral as the outer function.

c. *(2 marks)* Using part b. and the 2nd FTC determine $h'(x)$.

$$= -\int_{0}^{\arctan 3x} u\sqrt[3]{\sin u}\,du + \int_{0}^{\cot 2x} u\sqrt[3]{\sin u}\,du$$

$$= -f(g_1(x)) + f(g_2(x)) \quad \text{where} \quad f(x)=\int_0^x u\sqrt[3]{\sin u}\,du$$

c) $h'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x)$

$$= \frac{-3\arctan 3x\sqrt[3]{\sin \arctan 3x}}{1+(3x)^2}$$

$$-2\cot 2x\sqrt[3]{\sin(\cot(2x))}\,\csc^2(2x)$$

$g_1(x) = \arctan 3x$

$g_2(x) = \cot 2x$

and $f'(x) = x\sqrt[3]{\sin x}$

$g_1'(x) = \dfrac{1}{1+(3x)^2}\cdot 3$

$g_2'(x) = -\csc^2(2x)\cdot 2$

**Question 6.** Given

$$f(x)=e^{3x},\quad [0,\ln 2]$$

a. *(2 marks)* Find the average value of $f$ on the given interval.

b. *(2 marks)* Find $c$ such that $f_{ave}=f(c)$.

c. *(1 mark)* Sketch the graph of $f$ and a rectangle whose area is the same as the area under the graph of $f$.

a)

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{1}{\ln 2 - 0}\int_0^{\ln 2} e^{3x}dx$$

$$= \frac{1}{\ln 2}\int_0^{\ln 2} e^{3x}dx$$

$$= \frac{1}{\ln 2}\left[\frac{e^{3x}}{3}\right]_0^{\ln 2}$$

$$= \frac{1}{\ln 2}\left[\frac{e^{3\ln 2}}{3} - \frac{e^{3(0)}}{3}\right]$$

$$= \frac{1}{\ln 2}\left[\frac{e^{\ln 8}}{3} - \frac{1}{3}\right]$$

$$= \frac{7}{3\ln 2} \doteq 3.37$$

b)

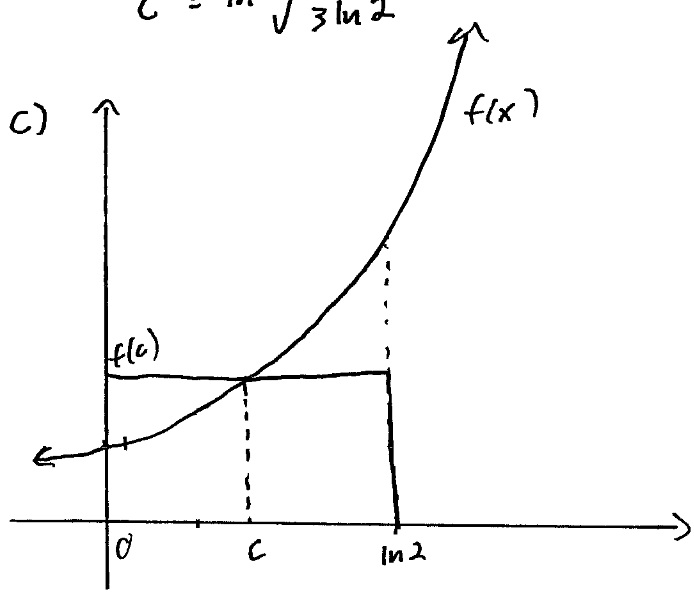
$$f_{ave} = f(c)$$

$$\frac{7}{3\ln 2} = e^{3c}$$

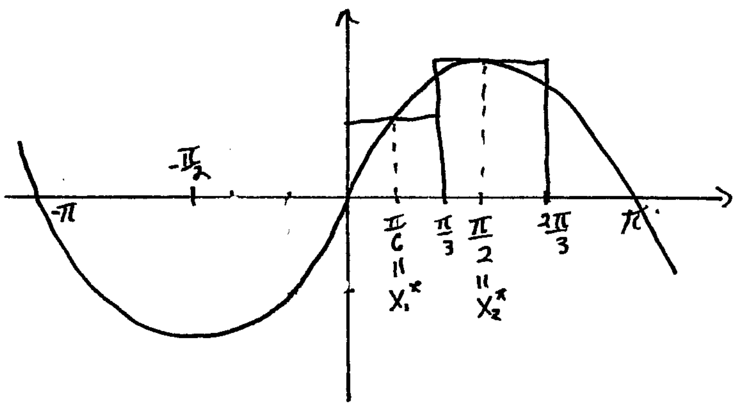
$$\ln\left(\frac{7}{3\ln 2}\right) = 3c$$

$$c = \ln\sqrt[3]{\frac{7}{3\ln 2}} \doteq 0.40$$

c)



**Question 7.** *(5 marks)* Estimate the definite integral of $f(x) = \sin x$ from $x = 0$ to $x = 2\pi/3$ using 2 rectangles and using the midpoints. Sketch the curve and the approximating rectangles.




$$\Delta x = \frac{b-a}{n} = \frac{\frac{2\pi}{3}-0}{2} = \frac{\pi}{3}$$

$$\int_0^{2\pi/3} \sin x \, dx \approx \sum_{i=1}^{2} f(x_i^*)\Delta x$$

$$= f(x_1^*)\Delta x + f(x_2^*)\Delta x$$

$$= \left(\sin\frac{\pi}{6}\right)\frac{\pi}{3} + \left(\sin\frac{\pi}{2}\right)\frac{\pi}{3}$$

$$= \left(\frac{1}{2} + 1\right)\frac{\pi}{3}$$

$$= \frac{\pi}{2}$$

**Question 8.** *(5 marks)* If $f$ is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(-x)\,dx = \int_{-b}^{-a} f(x)\,dx$$

For the case where $f(x) \geq 0$ and $0 < a < b$, draw a diagram to interpret this equation geometrically as an equality of areas.

$$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du)$$

$$= \int_{-b}^{-a} f(u)\,du$$

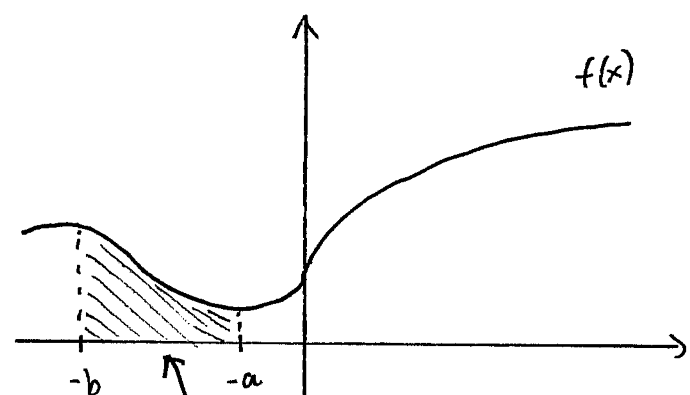
$u = -x$

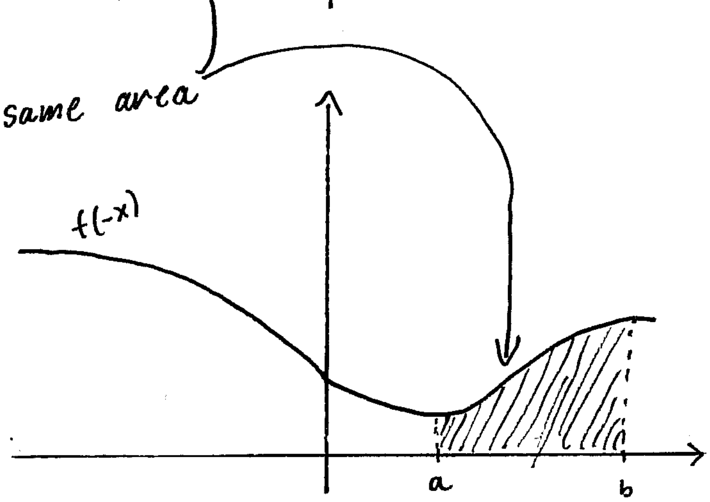
$du = -dx$

$-du = dx$

$u(b) = -b$

$u(a) = -a$



**Question 9.** *(3 marks)* Evaluate the definite integral

$$\int_{-5}^{1} \ln(4-3x)\,dx = \int_{19}^{1} \ln w \,\frac{dw}{-3} = \frac{-1}{3}\int_{19}^{1} \ln w \,dw$$

$w = 4-3x$

$dw = -3dx$

$\frac{dw}{-3} = dx$

$w(1) = 4-3(1) = 1$

$w(-5) = 4-3(-5) = 19$

$$= \frac{-1}{3}\left[\Big[uv\Big]_{19}^{1} - \int_{19}^{1} v\,du\right]$$

$u = \ln w$ $\quad du = \frac{1}{w}dw$

$v = w$ $\quad dv = dw$

$$= \frac{-1}{3}\left[\Big[w \ln w\Big]_{19}^{1} - \int_{19}^{1} \cancel{w}\,\frac{1}{\cancel{w}}\,dw\right]$$

$$= \frac{-1}{3}\left[1\ln 1 - 19\ln 19 - \Big[w\Big]_{19}^{1}\right]$$

$$= \frac{-19}{3}\ln 19 + \frac{1}{3}\left[1-19\right]$$

$$= \frac{-19}{3}\ln 19 + \frac{1}{3}\left[1-19\right]$$

$$= \frac{-19}{3}\ln 19 - 6$$

**Bonus Question.** *(3 marks)*
The Fresnel function is defined as

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)\,dt.$$

At what values of $x$ does this function have local maximum values, Justify.

① Find the critical points

$0 = S'(x)$

$0 = \frac{d}{dx}\left[\int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt\right]$

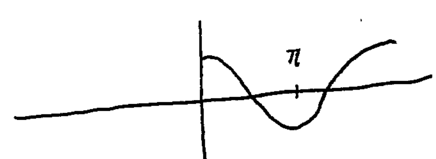
$0 = \sin\left(\frac{\pi x^2}{2}\right)$ by $2^{nd}$ FtC

$x = \pm\sqrt{2K}$ where $K \in \mathbb{Z}^+$

② Verity local max by using $2^{nd}$ derivative test

$S''(x) = \frac{d}{dx}\left[\sin\left(\frac{\pi x^2}{2}\right)\right]$

$= \cos\left(\frac{\pi x^2}{2}\right)\cdot \pi x$




if $x > 0$ then local max when $K$ is odd
if $x < 0$ then local max when $K$ is even.