Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S09: Winter 2013

Name: Y. Lamontagne
Student ID:

Test 1

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Evaluate the definite integral of $f(x) = -6x^2+3$ on $[-1,1]$ using the definition of the definite integral.

$$\int_{-1}^{1} f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{where } \Delta x = \frac{b-a}{n} = \frac{1-(-1)}{n} = \frac{2}{n}$$

$$x_i = -1 + i\frac{2}{n} = -1 + \frac{2i}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[-6\left[-1+\frac{2i}{n}\right]^2+3\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[-6\left(1-\frac{4i}{n}+\frac{4i^2}{n^2}\right)+3\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[-24\frac{i^2}{n^2}+24\frac{i}{n}-3\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^{n}-24\frac{i^2}{n^2}+\sum_{i=1}^{n}24\frac{i}{n}-\sum_{i=1}^{n}3\right]$$

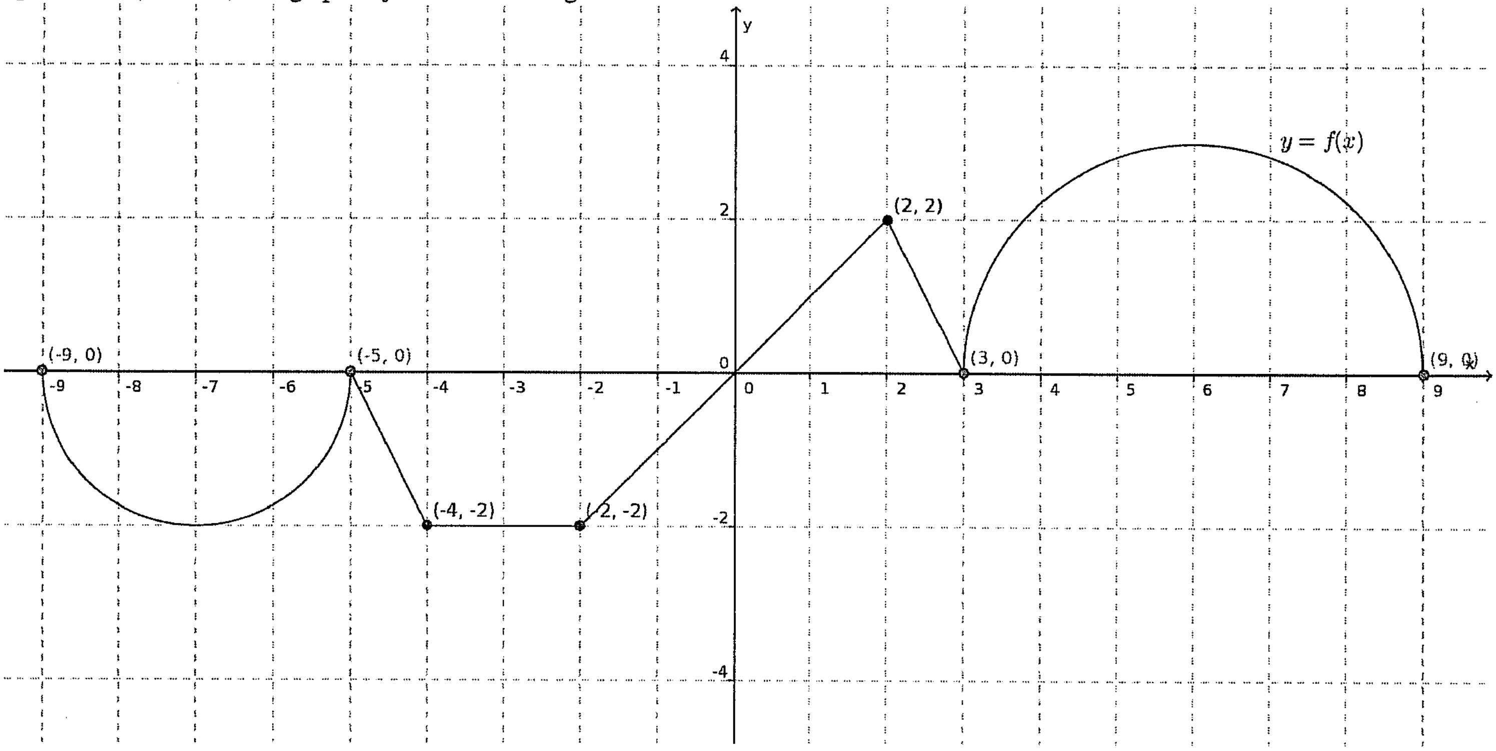
$$= \lim_{n\to\infty}\frac{2}{n}\left[\frac{-24}{n^2}\sum_{i=1}^{n}i^2+\frac{24}{n}\sum_{i=1}^{n}i-3n\right]$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[-\frac{\cancel{24}^{4}}{n^{\cancel{2}}}\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}}+\frac{\cancel{24}^{12}\cancel{n}(n+1)}{\cancel{n}\ \cancel{2}}-\frac{6n}{n}\right]$$

$$= \lim_{n\to\infty}\left[-\frac{8(n+1)}{n}\cdot\frac{(2n+1)}{n}+\frac{24(n+1)}{n}-6\right]$$

$$= -8\cdot 2+24-6 = 2$$

Question 2. *(5 marks)* The graph of f consists of straight lines and two semicircles.



Use the graph of the find the exact value of the following integrals.

a. *(1 mark)*

$$\int_2^6 f(x)\,dx = \frac{1(2)}{2} + \frac{\pi (3)^2}{4} = 1 + \frac{9\pi}{4}$$

b. *(1 mark)*

$$\int_{-2}^2 f(x)\,dx = 0$$

c. *(1 mark)*

$$\int_{-5}^{-3} f(x)\,dx = \frac{1(-2)}{2} + 1(-2) = -3$$

d. *(1 mark)*

$$\int_{-2}^6 f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_2^6 f(x)\,dx = 0 + 1 + \frac{9\pi}{4} = 1 + \frac{9\pi}{4}$$

e. *(1 mark)*

$$\int_{-7}^{-5} f(x)\,dx = -\frac{\pi 2^2}{4} = -\pi$$

Question 3. *(5 marks)* Evaluate the indefinite integral:

$$\int x^5\sqrt{x^3+e}\,dx = \int x^3\sqrt{x^3+e}\;x^2dx = \int (u-e)\sqrt{u}\,\frac{du}{3}$$

$u = x^3+e$

$du = 3x^2dx$

$\frac{du}{3} = x^2dx$

$u-e = x^3$

$$= \frac{1}{3}\int u^{3/2} - eu^{1/2}\,du$$

$$= \frac{1}{3}\left[\frac{2u^{5/2}}{5} - \frac{e2u^{3/2}}{3}\right] + C$$

$$= \frac{2}{15}(x^3+e)^{5/2} - \frac{2e}{9}(x^3+e)^{3/2} + C$$

Question 4. Given

$$h(x) = \int_{\operatorname{arcsec}2x}^{\tan 3x} u\sqrt[3]{\sin u}\,du = \int_{\operatorname{arcsec}2x}^{0} u\sqrt[3]{\sin u}\,du + \int_0^{\tan 3x} u\sqrt[3]{\sin u}\,du$$

a. *(2 marks)* Rewrite $h(x)$ as the sum of two integrals with a constant as the lower bound.

b. *(1 mark)* Rewrite the two integrals of part a. as composite functions with an integral as the outer function.

c. *(2 marks)* Using part b. and the 2nd FTC determine $h'(x)$.

$$= -\int_0^{\operatorname{arcsec}2x} u\sqrt[3]{\sin u}\,du + \int_0^{\tan 3x} u\sqrt[3]{\sin u}\,du$$

$$= -f(g_1(x)) + f(g_2(x))$$

where $f(x) = \int_0^x u\sqrt[3]{\sin u}\,du$, $g_1(x) = \operatorname{arcsec}2x$, $g_1'(x) = \frac{1}{2x\sqrt{(2x)^2-1}}\cdot 2$

$g_2(x) = \tan 3x$, $g_2'(x) = \sec^2 3x \cdot 3$

$$h'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x)$$

$$= \frac{-2\operatorname{arcsec}3x\sqrt[3]{\sin \operatorname{arcsec}3x}}{2x\sqrt{4x^2-1}} + 3\tan 3x\sqrt[3]{\sin\tan 3x}\,\sec^2 3x$$

Question 5. Given

$$f(x) = e^{2x}, \quad [0, \ln 3]$$

a. *(2 marks)* Find the average value of f on the given interval.

b. *(2 marks)* Find c such that $f_{ave} = f(c)$.

c. *(1 mark)* Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

a) $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{1}{\ln 3 - 0}\int_0^{\ln 3} e^{2x}\,dx$

$= \frac{1}{\ln 3}\int_0^{\ln 9} e^{u}\,\frac{du}{2}$

$= \frac{1}{2\ln 3}\left[e^{u}\right]_0^{\ln 9}$

$= \frac{1}{2\ln 3}\left[e^{\ln 9} - e^{0}\right]$

$= \frac{9-1}{2\ln 3}$

$= \frac{8}{2\ln 3} = \frac{4}{\ln 3} \doteq 3.64$

$u = 2x$

$du = 2\,dx$

$\frac{du}{2} = dx$

$u(0) = 2(0) = 0$

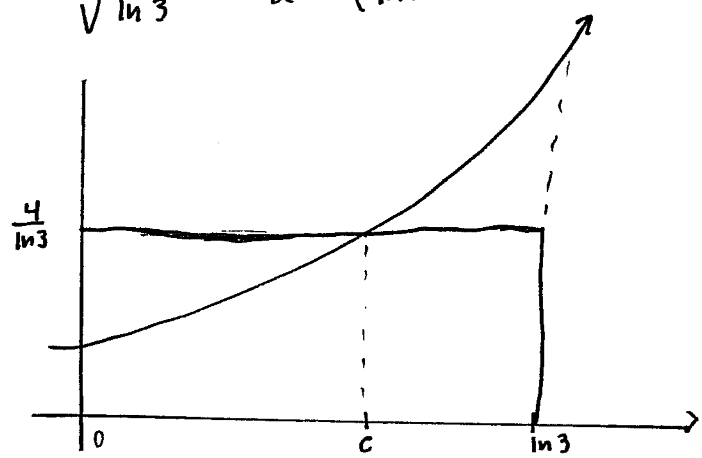
$u(\ln 3) = 2\ln 3 = \ln 9$

b) $\frac{4}{\ln 3} = e^{2c}$

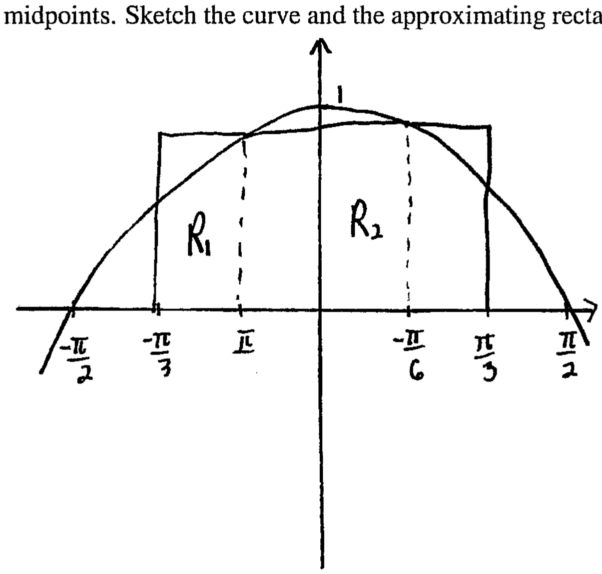
$\ln\left(\frac{4}{\ln 3}\right) = \ln e^{2c}$

$\ln\left(\frac{4}{\ln 3}\right) = 2c$

$0.65 \doteq \ln\sqrt{\frac{4}{\ln 3}} = \frac{1}{2}\ln\left(\frac{4}{\ln 3}\right) = c$



Question 6. *(5 marks)* Estimate the definite integral of $f(x) = \cos x$ from $x = -\pi/3$ to $x = \pi/3$ using two rectangles and using the midpoints. Sketch the curve and the approximating rectangles.



$\Delta x = \frac{b-a}{n} = \frac{\pi/3 - (-\pi/3)}{2} = \frac{2\pi/3}{2} = \frac{\pi}{3}$

$x_i = a + i\Delta x = -\frac{\pi}{3} + i\frac{\pi}{3}$

$x_0 = -\frac{\pi}{3}$

$x_1 = -\frac{\pi}{3} + 1\frac{\pi}{3} = 0$

$x_2 = -\frac{\pi}{3} + \frac{2\pi}{3} = \frac{\pi}{3}$

$x_1^* = \frac{x_0 + x_1}{2} = \frac{-\frac{\pi}{3} + 0}{2} = -\frac{\pi}{6}$

$x_2^* = \frac{x_1 + x_2}{2} = \frac{0 + \frac{\pi}{3}}{2} = \frac{\pi}{6}$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} f(x)\,dx \approx \sum_{i=1}^{2} f(x_i^*)\Delta x = \Delta x\left[f(x_1^*) + f(x_2^*)\right]$$

$$= \frac{\pi}{3}\left[f\left(-\frac{\pi}{6}\right) + f\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{3}\left[\cos\left(-\frac{\pi}{6}\right) + \cos\frac{\pi}{6}\right]$$

$$= \frac{\pi}{3}\left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] = \frac{\pi\sqrt{3}}{3}$$

Question 7. *(5 marks)* If f is continuous on $\mathbb{R}$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx.$$

For the case where $f(x) \geq 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

$$LHS = \int_a^b f(x+c)\,dx$$

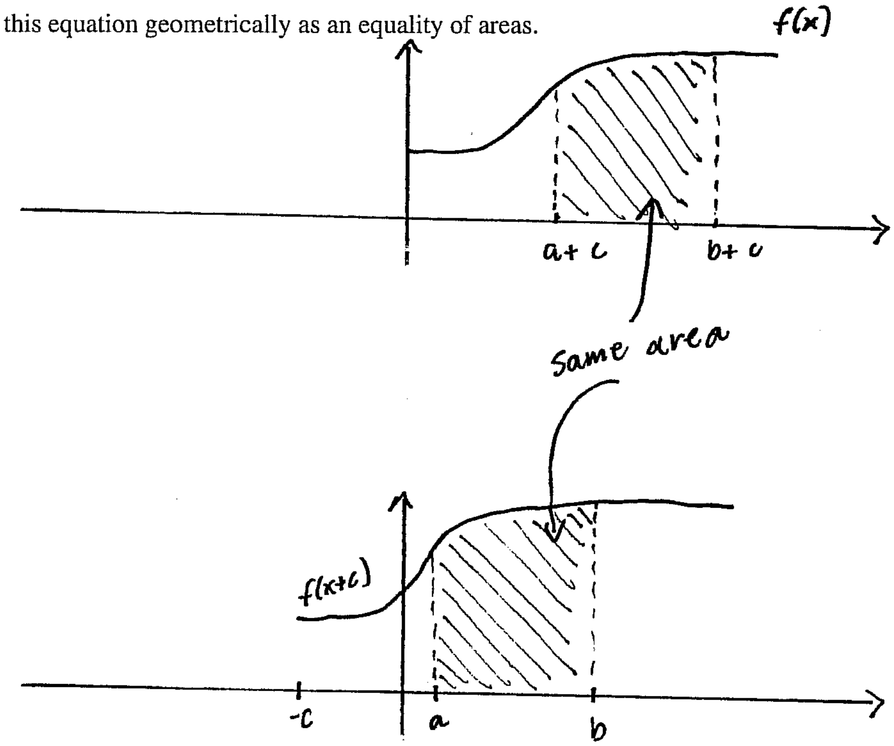
$u = x + c$

$du = dx$

$u(b) = b + c$

$u(a) = a + c$

$$= \int_{a+c}^{b+c} f(u)\,du$$



Question 8. *(5 marks)* Evaluate the definite integral

$$\int_{-3}^{1} \ln(3-2x)\,dx = \Big[uv\Big]_{-3}^{1} - \int_{-3}^{1} v\,du$$

$u = \ln(3-2x)$ $\qquad du = \frac{1}{3-2x} \cdot -2\,dx$

$v = x$ $\qquad dv = dx$

$$= \Big[x \ln(3-2x)\Big]_{-3}^{1} - \int_{-3}^{1} \frac{2x}{2x-3}\,dx$$

$$= 1\ln(3-2(1)) - (-3)\ln(3-2(-3)) - \int_{-3}^{1} \frac{2x-3+3}{2x-3}\,dx$$

$$= 3\ln 9 - \int_{-3}^{1} \frac{2x-3}{2x-3} + \frac{3}{2x-3}\,dx$$

$u = 2x - 3$ $\qquad u(1) = 2(1) - 3 = -1$

$du = 2\,dx$ $\qquad u(-3) = 2(-3) - 3 = -9$

$\frac{du}{2} = dx$

$$= 3\ln 9 - \left[\int_{-3}^{1} 1\,dx + 3\int_{-9}^{-1} \frac{1}{u}\frac{du}{2}\right]$$

$$= 3\ln 9 - \left[\Big[x\Big]_{-3}^{1} + \Big[\frac{3}{2}\ln|u|\Big]_{-9}^{-1}\right]$$

$$= 3\ln 9 - \left[\big[1-(-3)\big] + \frac{3}{2}\big[\ln|-1| - \ln|-9|\big]\right]$$

$$= 3\ln 9 - 4 + \frac{3}{2}\ln 9$$

$$= \frac{9}{2}\ln 9 - 4$$

Question 9. *(5 marks)* If

$$\int_2^7 (f(x)-3)\,dx = 5, \quad \int_2^0 3f(x)\,dx = 2,$$

and $f(x) = f(-x)$ for all $x \in \mathbb{R}$ find

$$\int_{-7}^7 f(x)\,dx. = 2\int_0^7 f(x)dx$$

$$= 2\left[\int_0^2 f(x)dx + \int_2^7 f(x)dx\right]$$

$$= 2\left[\frac{-2}{3} + 20\right]$$

$$= 2 \cdot \frac{58}{3}$$

$$= \frac{116}{3}$$

① $\int_2^0 3f(x)\,dx = 2$

$3\int_0^2 f(x)dx = -2$

$\int_0^2 f(x) = -\frac{2}{3}$

② $\int_2^7 (f(x)-3)dx = 5$

$\int_2^7 f(x)dx - \int_2^7 3\,dx = 5$

$\int_2^7 f(x)dx - \left[3x\right]_2^7 = 5$

$\int_2^7 f(x)dx = 5 + \left[3x\right]_2^7$

$= 5 + \left[3(7) - 3(2)\right]$

$= 5 + [21 - 6]$

$= 20$

③ If $x<0$ then local minimum when K is odd for the same argument as ② but since the integral is from larger lower bound to "smaller upper bound the "area" is of opposite sign as in ②.

② if $x>0$ then local minimum when K is even. since $\sin\left(\frac{\pi x^2}{2}\right)$ is alternating on the intervals generated by $\ast$. In addition the first interval on both sides of the y-axis is positive. Hence $S(x)$ is initially increasing if $x>0$.

Bonus Question. *(3 marks)*
The Fresnel function is defined as

$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right).$$

At what values of x does this function have local minimum values, Justify.

① Critical values when

$S'(x) = 0$

$0 = \sin\left(\frac{\pi x^2}{2}\right)$ by 2^{nd} FTC

$\ast$ $x = \pm\sqrt{2K}$ where $k \in \mathbb{Z}^+$

since