Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S09: Winter 2013

Name: Y. Lamontagne
Student ID: ________________

Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Evaluate the definite integral

$$\int_{\ln \sqrt[3]{\pi/12}}^{\ln \sqrt[3]{\pi/2}} e^{3x} \cos^2 e^{3x}\, dx = \int_{\pi/12}^{\pi/2} \cos^2 u \,\frac{du}{3} = \frac{1}{3}\int_{\pi/12}^{\pi/2} \frac{1+\cos 2u}{2}\, du$$

$u = e^{3x}$

$du = e^{3x}\, 3dx$

$\frac{du}{3} = e^{3x}\, dx$

$u(\ln \sqrt[3]{\pi/2}) = e^{3\ln\sqrt[3]{\pi/2}} = \pi/2$

$u(\ln \sqrt[3]{\pi/12}) = e^{3\ln\sqrt[3]{\pi/12}} = \pi/12$

$$= \frac{1}{6}\left[u + \frac{\sin 2u}{2} \right]_{\pi/12}^{\pi/2}$$

$$= \frac{1}{6}\left[\left[\pi/2 + \frac{\sin 2\frac{\pi}{2}}{2} \right] - \left[\pi/12 + \frac{\sin 2\frac{\pi}{12}}{2} \right] \right]$$

$$= \frac{1}{6}\left[\frac{\pi}{2} - \frac{\pi}{12} - \frac{1}{2}\sin\frac{\pi}{6} \right]$$

$$= \frac{1}{6}\left[\frac{5\pi}{12} - \frac{1}{4} \right]$$

Question 2. *(5 marks)* Evaluate the indefinite integral

$$\int \frac{x^3+x^2+1}{x^3+x^2+2x}\,dx = \int \frac{x^3+x^2+2x-2x+1}{x^3+x^2+2x}\,dx = \int \cancel{\frac{x^3+x^2+2x}{x^3+x^2+2x}} + \frac{-2x+1}{x^3+x^2+2x}\,dx$$

$$= \int 1 + \frac{-2x+1}{x(x^2+x+2)}\,dx$$

$$= x + \int \frac{-2x+1}{x(x^2+x+2)}\,dx$$

$$\frac{-2x+1}{x(x^2+x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+2}$$

$$-2x+1 = A(x^2+x+2) + (Bx+c)x$$

Let $x=0$

$$1 = 2A$$

$$\tfrac{1}{2} = A$$

So $-2x+1 = \frac{1}{2}x^2 + \frac{1}{2}x + 1 + Bx^2 + Cx$

$$0 = \left(\tfrac{1}{2}+B\right)x^2 + \left(\tfrac{5}{2}+C\right)x$$

$\therefore$ $B = \frac{-1}{2}$, $C = \frac{-5}{2}$

$$= x + \int \frac{1/2}{x} + \frac{-1/2x+2}{x^2+x+2}\,dx$$

$$= x + \tfrac{1}{2}\ln|x| + \int \frac{-1/2x - 5/2}{x^2+x+2}\,dx$$

$$= x + \tfrac{1}{2}\ln|x| + \int \frac{-1/2x - 1/4 - 9/4}{x^2+x+2}\,dx$$

$u = x^2+x+2$

$du = (2x+1)\,dx$

$\frac{du}{-4} = \left(\frac{-1}{2}x - \frac{1}{4}\right)dx$

$$= x + \tfrac{1}{2}\ln|x| + \int \frac{-1/2x - 1/4}{x^2+x+2}\,dx - \int \frac{9/4}{x^2+x+2}\,dx$$

$$= x + \tfrac{1}{2}\ln|x| + \int \frac{1}{u}\frac{du}{-4} - \frac{9}{4}\int \frac{1}{(x+\frac{1}{2})^2+\frac{7}{4}}\,dx$$

x^2+x+2

$= x^2+x+\frac{1}{4}-\frac{1}{4}+2$

$= (x+\frac{1}{2})^2 + \frac{7}{4}$

$$= x + \tfrac{1}{2}\ln|x| - \tfrac{1}{4}\ln|u| - \frac{9}{4}\frac{1}{\sqrt{\frac{7}{4}}}\arctan\left(\frac{(x+\frac{1}{2})}{\sqrt{\frac{7}{4}}}\right) + C$$

$$= x + \tfrac{1}{2}\ln|x| - \tfrac{1}{4}\ln(x^2+x+2) - \frac{9}{2\sqrt{7}}\arctan\left(\frac{2x+1}{\sqrt{7}}\right) + C$$

Question 3. *(5 marks)* Find the exact length of the curve

$$y = \ln(\cos x), \quad 0 \leq x \leq \frac{\pi}{3}$$

$$S = \int_0^{\pi/3} \sqrt{1 + (y')^2}\, dx \qquad y' = \frac{1}{\cos x} \cdot -\sin x = -\tan x$$

$$= \int_0^{\pi/3} \sqrt{1 + (-\tan x)^2}\, dx$$

$$= \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/3} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/3} |\sec x|\, dx$$

$$= \int_0^{\pi/3} \sec x\, dx$$

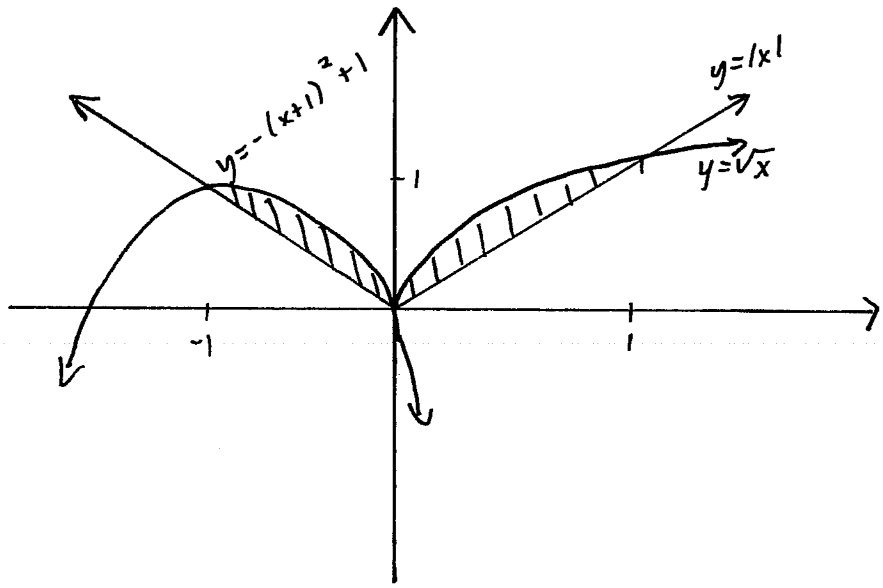
$$= \Big[\ln|\sec x + \tan x| \Big]_0^{\pi/3}$$

$$= \ln|\sec \pi/3 + \tan \pi/3| - \ln|\sec 0 + \tan 0|$$

$$= \ln|2 + \sqrt{3}|$$

Question 4. *(5 marks)* Sketch the region(s) enclosed by the given curves and set up the integral to find the total area of the region(s).

$y = |x|,\ y = \sqrt{x},\ y = -(x+1)^2 + 1$



$$A = \int_{-1}^{0} -(x+1)^2 + 1 - |x| \, dx + \int_{0}^{1} \sqrt{x} - |x| \, dx$$

Question 5. *(5 marks)* Determine whether the integral is convergent or divergent. Evaluate the integral if convergent.

$$\int_{\sqrt{3}}^{2\sqrt{3}} \frac{1}{x\sqrt{x^2-3}}\,dx$$ infinite discontinuity at $x = \sqrt{3}$

$$= \lim_{a \to \sqrt{3}^+} \int_a^{2\sqrt{3}} \frac{1}{x\sqrt{x^2-3}}\,dx$$

$$= \lim_{a \to \sqrt{3}^+} \left[\frac{1}{\sqrt{3}} \operatorname{arcsec} \frac{x}{\sqrt{3}} \right]_a^{2\sqrt{3}}$$

$$= \lim_{a \to \sqrt{3}^+} \left[\frac{1}{\sqrt{3}} \operatorname{arcsec} \frac{2\sqrt{3}}{\sqrt{3}} - \frac{1}{\sqrt{3}} \cancelto{0}{\operatorname{arcsec} \frac{a}{\sqrt{3}}} \right]$$

$$= \frac{1}{\sqrt{3}} \operatorname{arcsec} 2$$

$$= \frac{\pi}{3\sqrt{3}}$$

Question 6. *(5 marks)* Evaluate the definite integral

$$\int_3^5 \sqrt{x^2-2x-3}\,dx = \int_3^5 \sqrt{(x-1)^2-4}\,dx$$

$x-1 = 2\sec\theta$

$dx = 2\sec\theta\tan\theta$

x^2-2x-3

$= x^2-2x+1-1-3$

$= (x-1)^2-4$

$$= \int_0^{\frac{\pi}{3}} \sqrt{(2\sec\theta)^2-4}\;2\sec\theta\tan\theta\,d\theta$$

$5-1 = 2\sec\theta$

$2 = \sec\theta \Rightarrow \theta = \frac{\pi}{3}$

$3-1 = 2\sec\theta$

$1 = \sec\theta \Rightarrow \theta = 0$

$$= 2\int_0^{\pi/3} \sqrt{4\sec^2\theta-4}\;\sec\theta\tan\theta\,d\theta$$

$$= 2\int_0^{\pi/3} \sqrt{4(\sec^2\theta-1)}\;\sec\theta\tan\theta\,d\theta$$

$$= 2\int_0^{\pi/3} \sqrt{4\tan^2\theta}\;\sec\theta\tan\theta\,d\theta$$

$$= 4\int_0^{\pi/3} \sec\theta\tan^2\theta\,d\theta$$

$$= 4\int_0^{\pi/3} \sec\theta(\sec^2\theta-1)\,d\theta$$

$$= 4\int_0^{\pi/3} \sec^3\theta\,d\theta - 4\int_0^{\pi/3} \sec\theta\,d\theta$$

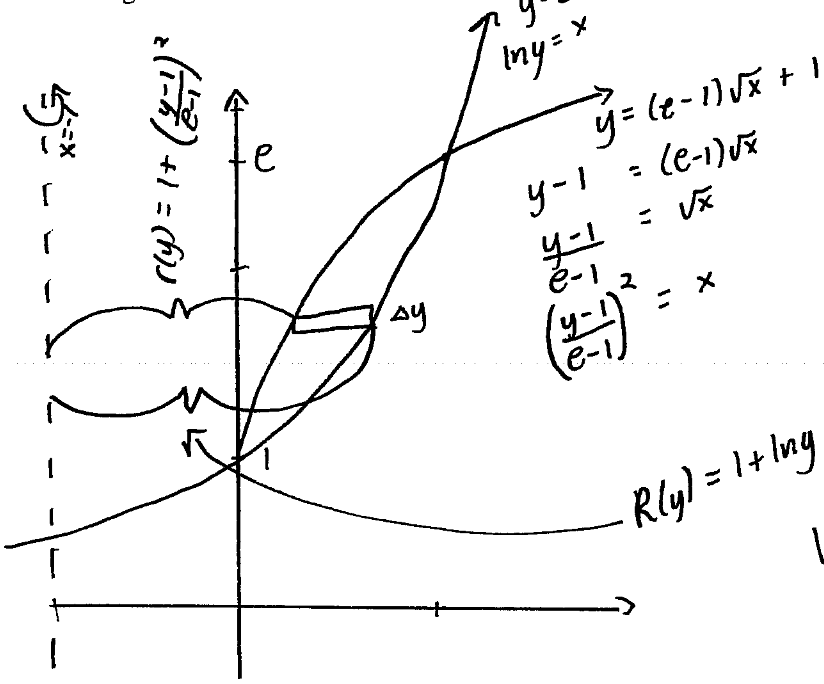
$$= 4\left[\sqrt{3}+\frac{1}{2}\ln(2+\sqrt{3})\right] - 4\Big[\ln|\sec\theta+\tan\theta|\Big]_0^{\pi/3}$$

$$= 4\sqrt{3}+2\ln(2+\sqrt{3}) - 4\left[\ln|\sec\tfrac{\pi}{3}+\tan\tfrac{\pi}{3}| - \ln|\sec 0+\tan 0|\right]$$

$$= 4\sqrt{3}+2\ln(2+\sqrt{3}) - 4\ln(2+\sqrt{3})$$

$$= 4\sqrt{3} - 2\ln(2+\sqrt{3})$$

Question 7. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $y = (e-1)\sqrt{x}+1$, $y = e^x$ rotated about the line $x = -1$. Sketch the region, draw a representative rectangle, write a representative element and the integral.



$$\Delta V = \pi\left[R(y)^2 - r(y)^2\right]\Delta y$$

$$= \pi\left[(1+\ln y)^2 - \left(1+\left(\frac{y-1}{e-1}\right)^2\right)^2\right]\Delta y$$

$$V = \int_1^e \pi\left[(1+\ln y)^2 - \left(1+\left(\frac{y-1}{e-1}\right)^2\right)^2\right]dy$$

Question 8. *(5 marks)* Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{6x}{x^2+3} - \frac{C}{3x+2}\right)\,dx$$

converges. Evaluate the integral for this value of C.

$$= \lim_{b\to\infty} \int_0^b \frac{6x}{x^2+3} - \frac{C}{3x+2}\,dx$$

$$= \lim_{b\to\infty} \left[3\ln(x^2+3) - \frac{C}{3}\ln|3x+2| \right]_0^b$$

$$= \lim_{b\to\infty} \left[\left[3\ln(b^2+3) - \frac{C}{3}\ln|3b+2| \right] - \left[3\ln(0^2+3) - \frac{C}{3}\ln(0+2)\right]\right]$$

l.f. $\infty - \infty$ if $C>0$

$$= \lim_{b\to\infty} \ln(b^2+3)^3 - \ln(3b+2)^{C/3} - \ln\left(\frac{27}{\sqrt[3]{2^C}}\right)$$

$$= \lim_{b\to\infty} \ln\frac{(b^2+3)^3}{(3b+2)^{C/3}} - \ln\left(\frac{27}{\sqrt[3]{2}}\right)$$

if $C > 18$ then $\lim_{n\to\infty} \frac{(b^2+3)^3}{(3b+2)^{C/3}} = 0$ so the integral diverges to $-\infty$

if $C = 18$ then $\lim_{n\to\infty} \frac{(b^2+3)^3}{(3b+2)} = \frac{1}{3^6}$ so the integral converges to $\ln\left(\frac{1}{3^6}\right) - \ln\left(\frac{27}{\sqrt[3]{2^{18}}}\right)$

if $C < 18$ then $\lim_{n\to\infty} \frac{(b^2+3)^3}{(3b+2)} = \infty$ so the integral diverges to ∞.

Question 9. *(5 marks)* Evaluate the indefinite integral

$$\int \frac{dx}{x\sqrt{5-x^2}}$$

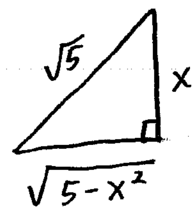
$x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta\, d\theta$

$$= \int \frac{\sqrt{5}\cos\theta\, d\theta}{\sqrt{5}\sin\theta\sqrt{5-(\sqrt{5}\sin\theta)^2}}$$

$\frac{opp}{hyp} = \frac{x}{\sqrt{5}} = \sin\theta$

$$= \int \frac{\cancel{\sqrt{5}}\cos\theta\, d\theta}{\cancel{\sqrt{5}}\sin\theta\sqrt{5-5\sin^2\theta}}$$



$$= \int \frac{\cos\theta}{\sin\theta\sqrt{5(1-\sin^2\theta)}}\, d\theta$$

$$= \int \frac{\cos\theta}{\sin\theta\sqrt{5\cos^2\theta}}\, d\theta$$

$$= \int \frac{\cancel{\cos\theta}}{\sin\theta\sqrt{5}\cancel{\cos\theta}}\, d\theta$$

$$= \frac{1}{\sqrt{5}}\int \csc\theta\, d\theta$$

$$= \frac{-1}{\sqrt{5}} \ln|\csc\theta + \cot\theta| + C$$

$$= \frac{-1}{\sqrt{5}} \ln\left|\frac{\sqrt{5}}{x} + \frac{\sqrt{5-x^2}}{x}\right| + C$$

Bonus Question. *(3 marks)* Find the exact length of the curve

$$y = \int_2^{\frac{e^x+1}{e^x-1}} \frac{1}{z}\, dz \quad [\ln 2, \ln 3]$$

$$S = \int_{\ln 2}^{\ln 3} \sqrt{1+(y')^2}\, dx$$

$$y' = \frac{1}{\frac{e^x+1}{e^x-1}} \cdot \frac{e^x(e^x-1)-(e^x+1)e^x}{(e^x-1)^2}$$

$$= \frac{-2e^x}{e^{2x}-1}$$

$$\therefore S = \int_{\ln 2}^{\ln 3} \sqrt{1+\left(\frac{-2e^x}{e^{2x}-1}\right)^2}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{1+\frac{4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{(e^{2x}-1)^2+4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\sqrt{e^{4x}+2e^{2x}+1}}{(e^{2x}-1)}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\sqrt{(e^{2x}+1)^2}}{(e^{2x}-1)}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x}+1}{e^{2x}-1}\, dx$$

add and remove

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x}+e^{2x}-e^{2x}+1}{e^{2x}-1}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{-e^{2x}+1+2e^{2x}}{e^{2x}-1}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{-e^{2x}+1}{e^{2x}-1}\, dx + \int_{\ln 2}^{\ln 3} \frac{2e^{2x}}{e^{2x}-1}\, dx$$

$$= -\int_{\ln 2}^{\ln 3} \frac{\cancel{e^{2x}-1}}{\cancel{e^{2x}-1}}\,dx + \int_{\ln 2}^{\ln 3} \frac{2e^{2x}}{e^{2x}-1}\,dx$$

$$= -\left[x\right]_{\ln 2}^{\ln 3} + \left[\ln|e^{2x}-1|\right]_{\ln 2}^{\ln 3}$$

$$= -\ln 3 + \ln 2 + \left[\ln|e^{2\ln 3}-1| - \ln|e^{2\ln 2}-1|\right]$$

$$= \ln\frac{2}{3} + \ln|e^{\ln 3^2}-1| - \ln|e^{\ln 2^2}-1|$$

$$= \ln\frac{2}{3} + \ln|9-1| - \ln|4-1|$$

$$= \ln\frac{2}{3} + \ln 8 - \ln 3$$

$$= \ln\frac{2}{3} + \ln\frac{8}{3}$$

$$= \ln\frac{2}{3}\cdot\frac{8}{3}$$

$$= \ln\frac{16}{9}$$