Dawson College: Calculus II (SCIENCE): 201-NYB-05-S09: Winter 2013

Name: Y. Lamontagne
Student ID: ________________

Test 2

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Evaluate the definite integral

$$\int_5^{\frac{6}{\sqrt{3}}+2} \sqrt{x^2-4x-5}\,dx = \int_5^{\frac{6}{\sqrt{3}}+2} \sqrt{(x-2)^2-9}\,dx$$

given that

$$\int_0^{\pi/6} \sec^3 x\,dx = \frac{1}{3}+\frac{\ln 3}{4}.$$

$x-2 = 3\sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$5-2 = 3\sec\theta$

$1 = \sec\theta \Rightarrow \theta = 0$

$\frac{6}{\sqrt{3}}+2-2 = 3\sec\theta$

$\frac{2}{\sqrt{3}} = \sec\theta \Rightarrow \theta = \frac{\pi}{6}$

$x^2-4x+4-4-5$

$= (x-2)^2-9$

$$= \int_0^{\pi/6} \sqrt{(3\sec\theta)^2-9}\;3\sec\theta\tan\theta\,d\theta$$

$$= \int_0^{\pi/6} \sqrt{9(\sec^2\theta-1)}\;3\sec\theta\tan\theta\,d\theta$$

$$= \int_0^{\pi/6} \sqrt{9\tan^2\theta}\;3\sec\theta\tan\theta\,d\theta$$

$$= 9\int_0^{\pi/6} \sec\theta\tan^2\theta\,d\theta$$

$$= 9\int_0^{\pi/6} \sec\theta(\sec^2\theta-1)\,d\theta$$

$$= 9\int_0^{\pi/6} \sec^3\theta - \sec\theta\,d\theta$$

$$= 9\int_0^{\pi/6} \sec^3\theta\,d\theta - 9\int_0^{\pi/6}\sec\theta\,d\theta$$

$$= 9\left[\frac{1}{3}+\frac{\ln 3}{4}\right] - 9\Big[\ln|\sec\theta+\tan\theta|\Big]_0^{\pi/6}$$

$$= 9\left[\frac{1}{3}+\frac{\ln 3}{4}\right] - 9\ln\left|\sec\frac{\pi}{6}+\tan\frac{\pi}{6}\right| + 9\ln|\sec 0+\tan 0|$$

$$= 9\left[\frac{1}{3}+\frac{\ln 3}{4}\right] - 9\ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)$$

Question 2. *(5 marks)* Determine whether the integral is convergent or divergent. Evaluate the integral if convergent.

$$\int_{\sqrt{5}}^{2\sqrt{5}} \frac{1}{x\sqrt{x^2-5}}\,dx = \lim_{a \to \sqrt{5}^+} \int_a^{2\sqrt{5}} \frac{1}{x\sqrt{x^2-5}}\,dx$$

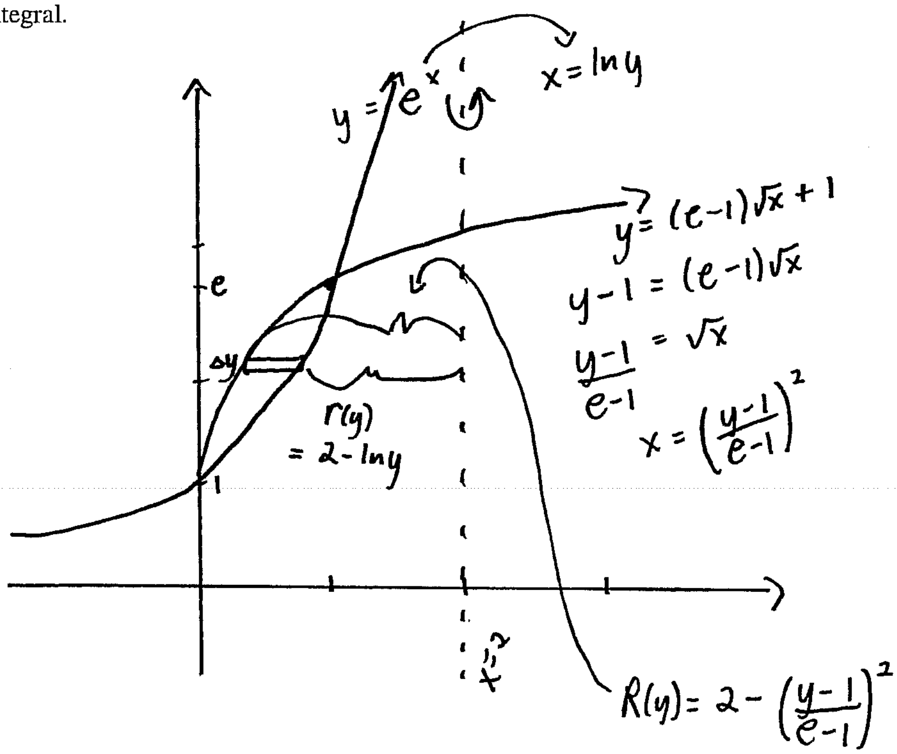
$$= \lim_{a \to \sqrt{5}^+} \left[\frac{1}{\sqrt{5}} \operatorname{arcsec} \frac{x}{\sqrt{5}}\right]_a^{2\sqrt{5}}$$

$$= \lim_{a \to \sqrt{5}^+} \frac{1}{\sqrt{5}}\left[\operatorname{arcsec} \frac{2\sqrt{5}}{\sqrt{5}} - \overbrace{\operatorname{arcsec} \frac{a}{\sqrt{5}}}^{0}\right]$$

$$= \frac{1}{\sqrt{5}} \operatorname{arcsec} 2$$

$$= \frac{\pi}{3\sqrt{5}}$$

Question 3. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $y = (e-1)\sqrt{x}+1$, $y = e^x$ rotated about the line $x = 2$. Sketch the region, draw a representative rectangle, write a representative element and the integral.



$$\Delta V = \pi\left[(R(y))^2 - (r(y))^2\right]\Delta y$$

$$= \pi\left[\left(2-\left(\frac{y-1}{e-1}\right)^2\right)^2 - (2-\ln y)^2\right]\Delta y$$

$$V = \int_1^e \pi\left[\left(2-\left(\frac{y-1}{e-1}\right)^2\right)^2 - (2-\ln y)^2\right]dy$$

Question 4. *(5 marks)* Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{8x}{x^2+4} - \frac{C}{4x+3}\right) dx$$

converges. Evaluate the integral for this value of C.

$$= \lim_{b\to\infty} \int_0^b \left(\frac{8x}{x^2+4} - \frac{c}{4x+3}\right)dx$$

$$= \lim_{b\to\infty} \left[4\ln(x^2+4) - \frac{c}{4}\ln|4x+3|\right]_0^b$$

$$= \lim_{b\to\infty} \left[\left[4\ln(b^2+4) - \frac{c}{4}\ln(4b+3)\right] - \left[4\ln 4 - \frac{c}{4}\ln 3\right]\right]$$

I.f. $\infty - \infty$ if. $c > 0$

$$= \lim_{b\to\infty} \left[\ln(b^2+4)^4 - \ln(4b+3)^{c/4} - \ln\left(\frac{256}{\sqrt[4]{3^c}}\right)\right]$$

$$= \lim_{b\to\infty} \ln\left(\frac{(b^2+4)^4}{(4b+3)^{c/4}}\right) - \ln\left(\frac{256}{\sqrt[4]{3^c}}\right)$$

if $c > 32$ then $\lim_{b\to\infty} \frac{(b^2+4)^4}{(4b+3)^{c/4}} = 0$ so the integral diverges to $-\infty$

if $c = 32$ then $\lim_{b\to\infty} \frac{(b^2+4)^4}{(4b+3)^8} = \frac{1}{4^8}$ so the integral converges to $\ln\left(\frac{1}{4^8}\right) - \ln\left(\frac{256}{\sqrt[4]{3^{32}}}\right)$

if $c < 32$ then $\lim_{b\to\infty} \frac{(b^2+4)^4}{(4b+3)^{c/4}} = \infty$ so the integral diverges to ∞.

Question 5. *(5 marks)* Evaluate the indefinite integral

$$\int \frac{x^3+x^2+2}{x^3+x^2+x}\,dx = \int \frac{x^3+x^2+x-x+2}{x^3+x^2+x}\,dx = \int \frac{x^3+x^2+x}{x^3+x^2+x} + \frac{-x+2}{x^3+x^2+x}\,dx$$

$$= \int 1 + \frac{-x+2}{x(x^2+x+1)}\,dx$$

$$\frac{-x+2}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$$-x+2 = A(x^2+x+1) + (Bx+C)x$$

Let $x=0$

$$2 = A$$

So

$$-x+2 = 2(x^2+x+1) + (Bx+C)x$$

$$0 = (2+B)x^2 + (3+C)$$

$$\therefore B=-2,\quad C=-3$$

$$= x + \int \frac{2}{x} + \frac{-2x-3}{x^2+x+1}\,dx$$

$$= x + 2\ln|x| + \int \frac{-2x-1-2}{x^2+x+1}\,dx$$

$$= x + 2\ln|x| + \int \frac{-2x-1}{x^2+x+1}\,dx - 2\int \frac{1}{x^2+x+1}\,dx$$

$$u = x^2+x+1$$

$$du = (2x+1)\,dx$$

$$-du = (-2x-1)\,dx$$

$$x^2+x+1$$

$$= x^2 + x + \frac{1}{4} - \frac{1}{4} + 1$$

$$= \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= x + 2\ln|x| + \int \frac{1}{u}(-du) - 2\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$$

$$= x + 2\ln|x| - \ln|u| - 2\frac{1}{\sqrt{\frac{3}{4}}}\arctan\left(\frac{(x+\frac{1}{2})}{\sqrt{\frac{3}{4}}}\right) + C$$

$$= x + 2\ln|x| - \ln(x^2+x+1)$$

$$- \frac{4}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Question 6. *(5 marks)* Evaluate the definite integral

$$\int_{\ln \sqrt[5]{\pi/6}}^{\ln \sqrt[5]{\pi/2}} e^{5x} \sin^2 e^{5x}\, dx \qquad = \int_{\pi/6}^{\pi/2} \sin^2 u \,\frac{du}{5}$$

$$u = e^{5x}$$

$$du = 5e^{5x}\,dx$$

$$\frac{du}{5} = e^{5x}\,dx$$

$$u\left(\ln \sqrt[5]{\pi/2}\right) = e^{5\ln\sqrt[5]{\pi/2}} = \frac{\pi}{2}$$

$$u\left(\ln \sqrt[5]{\pi/6}\right) = e^{5\ln\sqrt[5]{\pi/6}} = \pi/6$$

$$= \frac{1}{5}\int_{\pi/6}^{\pi/2} \frac{1-\cos 2u}{2}\, du$$

$$= \frac{1}{10}\left[\theta - \frac{\sin 2u}{2}\right]_{\pi/6}^{\pi/2}$$

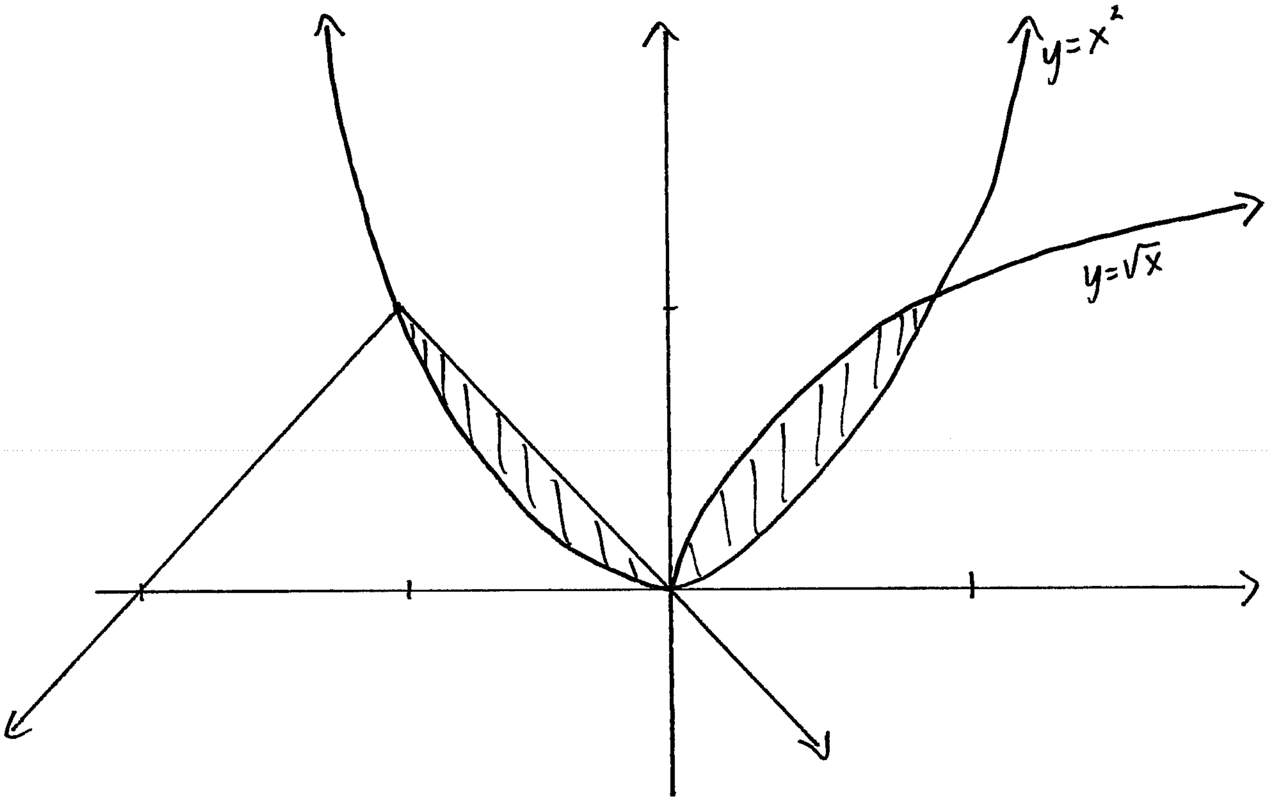
$$= \frac{1}{10}\left[\left[\pi/2 - \frac{\sin 2(\frac{\pi}{2})}{2}\right] - \left[\frac{\pi}{6} - \frac{\sin 2(\frac{\pi}{6})}{2}\right]\right]$$

$$= \frac{1}{10}\left[\pi/2 - \frac{\pi}{6} + \frac{\sin \pi/3}{2}\right]$$

$$= \frac{1}{10}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right]$$

Question 7. *(5 marks)* Sketch the region(s) enclosed by the given curves and set up the integral to find the total area of the region(s).

$y = -|x+1|+1,\ y = \sqrt{x},\ y = x^2$



$$A = \int_{-1}^{0} -|x+1|+1-x^2\,dx + \int_{0}^{1} \sqrt{x}-x^2\,dx$$

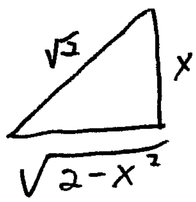
Question 8. *(5 marks)* Evaluate the indefinite integral

$$\int \frac{dx}{x\sqrt{2-x^2}}$$

$x = \sqrt{2}\sin\theta$

$dx = \sqrt{2}\cos\theta\, d\theta$

$\frac{x}{\sqrt{2}} = \sin\theta$



$$= \int \frac{\cancel{\sqrt{2}}\cos\theta}{\cancel{\sqrt{2}}\sin\theta\sqrt{2-(\sqrt{2}\sin\theta)^2}}\, d\theta$$

$$= \int \frac{\cos\theta}{\sin\theta\sqrt{2-2\sin^2\theta}}\, d\theta$$

$$= \int \frac{\cos\theta}{\sin\theta\sqrt{2(1-\sin^2\theta)}}\, d\theta$$

$$= \int \frac{\cos\theta}{\sin\theta\sqrt{2\cos^2\theta}}\, d\theta$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sin\theta}\, d\theta$$

$$= \frac{1}{\sqrt{2}}\int \csc\theta\, d\theta$$

$$= \frac{-1}{\sqrt{2}} \ln|\csc\theta + \cot\theta| + C$$

$$= \frac{-1}{\sqrt{2}} \ln\left|\frac{\sqrt{2}}{x} + \frac{\sqrt{2-x^2}}{x}\right| + C$$

Question 9. *(5 marks) (5 marks)* Find the exact length of the curve

$$y = \ln(\sec x), \quad 0 \le x \le \frac{\pi}{4}$$

$$y' = \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x$$

$$S = \int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\pi/4} |\sec x|\, dx$$

$$= \int_0^{\pi/4} \sec x dx$$

$$= \Big[\ln|\sec x + \tan x| \Big]_0^{\pi/4}$$

$$= \ln|\sec \pi/4 + \tan \pi/4| - \ln|\sec 0 + \tan 0|$$

$$= \ln(\sqrt{2} + 1)$$

Bonus Question. *(3 marks)* Find the exact length of the curve

$$y = \int_2^{\frac{e^x+1}{e^x-1}} \frac{1}{z}\, dz \quad [\ln 2, \ln 3]$$

$$S = \int_{\ln 2}^{\ln 3} \sqrt{1+(y')^2}\, dx$$

$$y' = \frac{1}{\frac{e^x+1}{e^x-1}} \cdot \frac{e^x(e^x-1)-(e^x+1)e^x}{(e^x-1)^2}$$

$$= \frac{-2e^x}{e^{2x}-1}$$

$$\therefore S = \int_{\ln 2}^{\ln 3} \sqrt{1+\left(\frac{-2e^x}{e^{2x}-1}\right)^2}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{1+\frac{4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{(e^{2x}-1)^2+4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \sqrt{\frac{e^{4x}-2e^{2x}+1+4e^{2x}}{(e^{2x}-1)^2}}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\sqrt{e^{4x}+2e^{2x}+1}}{(e^{2x}-1)}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{\sqrt{(e^{2x}+1)^2}}{(e^{2x}-1)}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x}+1}{e^{2x}-1}\, dx$$

add and remove

$$= \int_{\ln 2}^{\ln 3} \frac{e^{2x}+e^{2x}-e^{2x}+1}{e^{2x}-1}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{-e^{2x}+1+2e^{2x}}{e^{2x}-1}\, dx$$

$$= \int_{\ln 2}^{\ln 3} \frac{-e^{2x}+1}{e^{2x}-1}\, dx + \int_{\ln 2}^{\ln 3} \frac{2e^{2x}}{e^{2x}-1}\, dx$$

$$= -\int_{\ln 2}^{\ln 3} \frac{\cancel{e^{2x}-1}}{\cancel{e^{2x}-1}}\,dx + \int_{\ln 2}^{\ln 3} \frac{2e^{2x}}{e^{2x}-1}\,dx$$

$$= -\left[x\right]_{\ln 2}^{\ln 3} + \left[\ln|e^{2x}-1|\right]_{\ln 2}^{\ln 3}$$

$$= -\ln 3 + \ln 2 + \left[\ln|e^{2\ln 3}-1| - \ln|e^{2\ln 2}-1|\right]$$

$$= \ln\frac{2}{3} + \ln|e^{\ln 3^2}-1| - \ln|e^{\ln 2^2}-1|$$

$$= \ln\frac{2}{3} + \ln|9-1| - \ln|4-1|$$

$$= \ln\frac{2}{3} + \ln 8 - \ln 3$$

$$= \ln\frac{2}{3} + \ln\frac{8}{3}$$

$$= \ln\frac{2}{3}\cdot\frac{8}{3}$$

$$= \ln\frac{16}{9}$$