Dawson College: Calculus II (SCIENCE): 201-NYB-05-S09: Winter 2013

Name: Y. Laumontagne
Student ID: ______________

Test 3

test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)*

a. *(1 mark)* Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

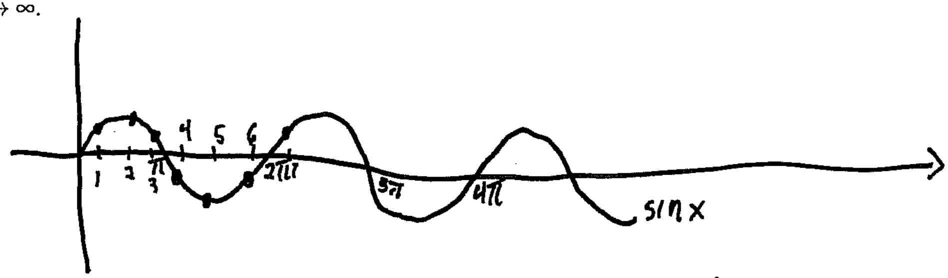
$$\left\{\sin 1, \frac{\sin 2}{2}, \frac{\sin 3}{3}, \frac{\sin 4}{4}, \frac{\sin 5}{5}, \ldots\right\} \qquad a_n = \frac{\sin n}{n}$$

b. *(1 mark)* Is a_n is monotonic?

c. *(1 mark)* Show that a_n is bounded.

d. *(2 marks)* Determine the limit of a_n as $n \to \infty$.

b)



The sequence is not monotonic since $a_4 < a_3$ and $a_7 > a_6$

c) Since $-1 < \sin n < 1$ the sequence is bounded by -1 and 1

d) $b_n = \frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} = c_n$

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$

$\therefore$ by the squeeze thm $a_n \to 0$ as $n \to \infty$

Question 2. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\pi n - \operatorname{arcsec} n}{\sqrt{n^7+n^3+1}}$$

Let $a_n = \dfrac{\pi n - \operatorname{arcsec} n}{\sqrt{n^7+n^3+1}}$

$$a_n = \frac{\pi n - \operatorname{arcsec} n}{\sqrt{n^7+n^3+1}} \le \frac{\pi n}{\sqrt{n^7+n^3+1}} \le \frac{\pi n}{\sqrt{n^7}} = \pi \frac{n}{n^{7/2}} = \pi \frac{1}{n^{5/2}} = b_n$$

$\sum\limits_{n=2}^{\infty} b_n$ is convergent since p-series where $p = 5/2 > 1$

$\therefore$ by the comparison test $\sum\limits_{n=2}^{\infty} a_n$ converges.

Question 3. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=3}^{\infty}(-1)^n\sqrt{\frac{n^2+n}{n^3-n-3}}$$

Lets determine whether the series is absolutely convergent.

$$\sum_{n=3}^{\infty}\left|(-1)^n\sqrt{\frac{n^2+n}{n^3-n-3}}\right| = \sum_{n=3}^{\infty}\sqrt{\frac{n^2+n}{n^3-n-3}} \qquad \text{Let } a_n = \sqrt{\frac{n^2+n}{n^3-n-3}}$$

$$a_n = \sqrt{\frac{n^2+n}{n^3-n-3}} \geq \sqrt{\frac{n^2}{n^3-n-3}} \geq \sqrt{\frac{n^2}{n^3-n}} \geq \sqrt{\frac{n^2}{n^3}} = \frac{1}{\sqrt{n}} = b_n$$

$\sum_{n=3}^{\infty} b_n$ is divergent since p-series where $p=\frac{1}{2}<1$

$\therefore \sum_{n=3}^{\infty} a_n$ is divergent by comparison test.

Lets determine whether the series is conditionally convergent.

$\sum_{n=3}^{\infty}(-1)^n b_n$ where $b_n = \sqrt{\frac{n^2+n}{n^3-n-3}}$, lets apply the alternating series test

① $b_{n+1} \leq b_n$: let $f(x) = \left(\frac{x^2+x}{x^3-x-3}\right)^{1/2}$

$$f'(x) = \frac{1}{2}\left(\frac{x^2+x}{x^3-x-3}\right)^{-1/2}\left[\frac{(2x+1)(x^3-x-3)-(3x^2-1)(x^2+x)}{(x^3-x-3)^2}\right]$$

$$= \frac{1}{2}\left(\frac{x^2+x}{x^3-x-3}\right)^{-1/2}\left[\frac{-x^4-2x^3-x^2-3}{(x^3-x-3)^2}\right] < 0$$

So $b_{n+1} \leq b_n$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\sqrt{\frac{n^2+n}{n^3-n-3}}$

$= 0$

$\therefore$ converges by alternating series test.

$\therefore$ conditionaly convergent.

Question 4. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{3^n (n!)^2}$$

Let $a_n = \dfrac{(-1)^n (2n)!}{3^n (n!)^2}$

Lets apply the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1} (2(n+1))!}{3^{n+1} ((n+1)!)^2}}{\dfrac{(-1)^n (2n)!}{3^n (n!)^2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (2n+2)!}{\cancel{3^n} 3 (\cancel{n!} (n+1))^2} \cdot \frac{\cancel{3^n} \cancel{(n!)^2}}{(-1)^n (2n)!} \right|$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1) \cancel{(2n)!}}{3(n+1)^2 \cancel{(2n)!}}$$

$$= \frac{4}{3} > 1$$

$\therefore$ diverges by ratio test.

Question 5. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=2}^{\infty} \frac{2^n - 3^{n-1}}{4^{n+1}}$$

Lets look at both series independently

$$\sum_{n=2}^{\infty} \frac{2^n}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{2^n}{4 4^n} = \sum_{n=2}^{\infty} \frac{1}{4}\left(\frac{2}{4}\right)^n = \sum_{n=2}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^n$$

conv. since geometric series where $r = \frac{1}{2} < 1$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^{n+2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{16}\left(\frac{1}{2}\right)^n = \frac{a}{1-r}$$

$$= \frac{1/16}{1-1/2} = 1/8$$

$$\sum_{n=2}^{\infty} \frac{3^{n-1}}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{3^{-1}3^n}{4 4^n} = \sum_{n=2}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^n$$

converges since geometric series where $r = \frac{3}{4} < 1$

$$= \sum_{n=0}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^{n+2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^2\left(\frac{3}{4}\right)^n = \frac{a}{1-r}$$

$$= \frac{\frac{1}{12}\left(\frac{3}{4}\right)^2}{1-\frac{3}{4}}$$

$$= \frac{4\,\frac{1}{\cancel{12}}\left(\frac{\cancel{9}\,3}{16}\right)}{\frac{1}{4}} = \frac{3}{16}$$

$$\sum_{n=2}^{\infty} \frac{2^n - 3^{n-1}}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{2^n}{4^{n+1}} - \sum_{n=2}^{\infty} \frac{3^{n-1}}{4^{n+1}}$$

$$= \frac{1}{8} - \frac{3}{16} = \frac{-1}{16}$$

Question 6. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1995}^{\infty} \frac{(-1)^n}{n(\ln n)^3}$$

Lets try to determine if the series is absolutely convergent.

$$\sum_{n=1995}^{\infty} \left| \frac{(-1)^n}{n(\ln n)^3} \right| = \sum_{n=1995}^{\infty} \frac{1}{n(\ln n)^3}$$

Let $f(x) = \frac{1}{x(\ln x)^3}$

- $f(x)$ is continuous for $x > 1995$
- $f(x)$ is positive for $x > 1995$
- $f'(x) = \frac{-1}{(x(\ln x)^3)^2} \cdot \left[(\ln x)^3 + x3(\ln x)^2 \frac{1}{x} \right] < 0$

$\therefore$ $f(x)$ is decreasing $x > 1995$.

$$\int_{1995}^{\infty} \frac{1}{x(\ln x)^3} dx = \lim_{b \to \infty} \int_{1995}^{b} \frac{1}{x(\ln x)^3} dx = \lim_{b \to \infty} \int_{\ln 1995}^{\ln b} \frac{1}{u^3} du$$

$u = \ln x$

$du = \frac{1}{x} dx$

$u(b) = \ln b$

$u(1995) = \ln 1995$

$$= \lim_{b \to \infty} \left[\frac{1}{-2u^2} \right]_{\ln 1995}^{\ln b}$$

$$= \lim_{b \to \infty} \left[\cancel{\frac{1}{-2(\ln b)^2}}^{\to 0} + \frac{1}{2(\ln 1995)^2} \right]$$

$$= \frac{1}{2(\ln 1995)^2}$$

$\therefore$ by the integral test the series converges

$\therefore$ the series is absolutely convergent

Question 7. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=3}^{\infty}\left[\left(\sin\left(\frac{\pi}{2}-\frac{1}{n}\right)\right)^{n^2}\right]^{n} \quad \text{Let} \quad a_n = \left[\left(\sin\left(\frac{\pi}{2}-\frac{1}{n}\right)\right)^{n^2}\right]^{n}$$

Lets apply the root test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\left[\left[\sin\left(\frac{\pi}{2}-\frac{1}{n}\right)\right]^{n^2}\right]^{n}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\sin\left(\frac{\pi}{2}-\frac{1}{n}\right)\right)^{n^2} \quad \text{I.f. } 1^{\infty}$$

$$y = \lim_{x\to\infty}\left(\sin\left(\frac{\pi}{2}-\frac{1}{x}\right)\right)^{x^2}$$

$$\ln y = \ln \lim_{x\to\infty}\left(\sin\left(\frac{\pi}{2}-\frac{1}{x}\right)\right)^{x^2}$$

$$\ln y = \lim_{x\to\infty}\ln\left(\sin\left(\frac{\pi}{2}-\frac{1}{x}\right)\right)^{x^2}$$

$$\ln y = \lim_{x\to\infty} x^2\ln\left(\sin\left(\frac{\pi}{2}-\frac{1}{x}\right)\right) \quad \text{I.f. } \infty\cdot 0$$

$$\ln y = \lim_{x\to\infty}\frac{\ln\left(\sin\left(\frac{\pi}{2}-\frac{1}{x}\right)\right)}{\frac{1}{x^2}} \quad \text{I.f. } \frac{0}{0}$$

$$\ln y \overset{\hat{H}}{=} \lim_{x\to\infty}\frac{\frac{1}{\sin\left(\frac{\pi}{2}-\frac{1}{x}\right)}\cos\left(\frac{\pi}{2}-\frac{1}{x}\right)\cdot\cancel{\frac{+1}{x^2}}}{\cancel{\frac{-2}{x^3}}\ \frac{2}{x}}$$

$$\ln y = \lim_{x\to\infty}\frac{-\cot\left(\frac{\pi}{2}-\frac{1}{x}\right)}{\frac{2}{x}} \quad \text{I.f. } \frac{0}{0}$$

$$\ln y \overset{\hat{H}}{=} \lim_{x\to\infty}\frac{+\csc^2\left(\frac{\pi}{2}-\frac{1}{x}\right)\cancel{\frac{+1}{x^2}}}{\cancel{\frac{-2}{x^2}}}$$

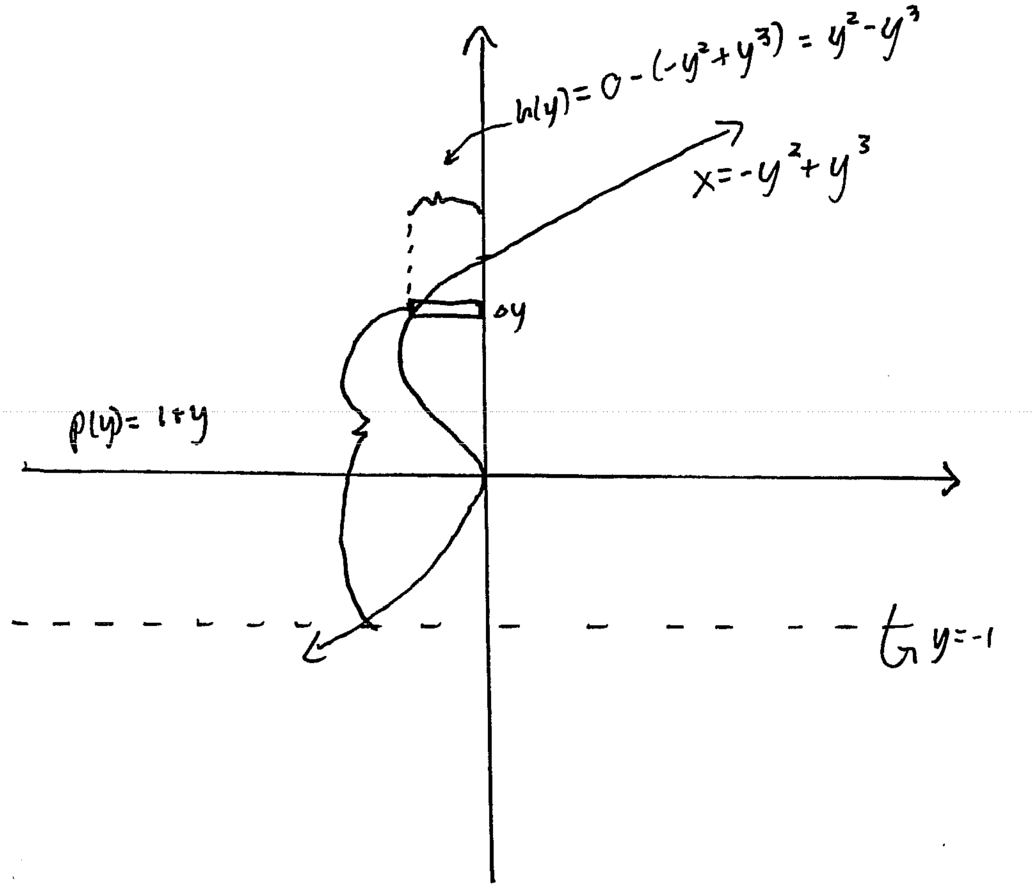
$$\ln y \overset{\hat{H}}{=} -\frac{1}{2}$$

$$y = e^{-1/2}$$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = e^{-1/2} < 1$$

$\therefore$ converges by root test.

Question 8. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $x = -y^2 + y^3$, $x = 0$ rotated about the line $y = -1$. Sketch the region, draw a representative rectangle, label all parts of the graph and give the integral.



y-int: $0 = -y^2 + y^3$

$0 = y^2(-1 + y)$

$y = 0 \qquad y = 1$

$$V = \int_0^1 2\pi\, p(y)\, h(y)\, dy$$

$$= \int_0^1 2\pi (1+y)(y^2 - y^3)\, dy$$

Question 9. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=2}^{\infty} [\ln(3n^3+1) - 3\ln n] = \sum_{n=2}^{\infty} [\ln(3n^3+1) - \ln n^3]$$

$$= \sum_{n=2}^{\infty} \ln\left(\frac{3n^3+1}{n^3}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{3n^3+1}{n^3}\right) = \ln 3 \neq 0$$

$\therefore$ does not converge by n^{th} term divergence test.

Bonus Question.

a. *(1 mark)* State the $K(\varepsilon)$ definition of the limit of a sequence.

b. *(3 marks)* Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$

c. *(2 marks)* If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1+a_n}$$

converges ~~[illegible]~~ then Find the limit of the sequence.

a) $\forall \varepsilon > 0 \ \exists K(\varepsilon) \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon \quad \forall n \geq K(\varepsilon)$

b) Given $\varepsilon > 0$ then $\exists K_1(\varepsilon)$ s.t. $|a_{2n} - L| < \varepsilon \quad \forall n \geq K_1(\varepsilon)$
then $\exists K_2(\varepsilon)$ s.t. $|a_{2n+1} - L| < \varepsilon \quad \forall n \geq K_2(\varepsilon)$

So let $K(\varepsilon) = \max\{2K_1(\varepsilon), 2K_2(\varepsilon)+1\}$ and it follows that if n is even or odd then

$$|a_n - L| < \varepsilon \quad \forall n \geq K(\varepsilon)$$

c) So $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_{n+1} = L$

$\therefore$ We have

$$a_{n+1} = 1 + \frac{1}{1+a_n}$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left[1 + \frac{1}{1+a_n}\right]$$

$$L = 1 + \frac{1}{1+L}$$

$$L - 1 = \frac{1}{1+L}$$

$$L^2 - 1 = 1$$

$$L^2 = 2$$

$$L = \pm\sqrt{2}$$

Since $a_n \geq 0$ then $L = \sqrt{2}$