Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S09: Winter 2013

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Test 3

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1995}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$$

Lets determine whether the series is absolutely convergent

$$\sum_{n=1995}^{\infty} \left| \frac{(-1)^n}{n(\ln n)^2} \right| = \sum_{n=1995}^{\infty} \frac{1}{n(\ln n)^2}$$

Let $f(x) = \frac{1}{x(\ln x)^2}$

- $f(x)$ is continuous for $x > 1995$
- $f(x)$ is positive for $x > 1995$
- $f'(x) = \frac{-1}{(x(\ln x)^2)^2} \cdot \left[(\ln x)^2 + x2(\ln x)\frac{1}{x} \right] < 0$ for $x > 1995$

$\therefore$ $f(x)$ is decreasing

$$\int_{1995}^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_{1995}^{b} \frac{1}{x(\ln x)^2}\,dx$$

$u = \ln x$

$du = \frac{1}{x} dx$

$u(b) = \ln b$

$u(1995) = \ln(1995)$

$$= \lim_{b\to\infty} \int_{\ln(1995)}^{\ln b} \frac{1}{u^2}\,du$$

$$= \lim_{b\to\infty} \left[\frac{-1}{u} \right]_{\ln 1995}^{\ln b}$$

$$= \lim_{b\to\infty} \left[\cancel{\frac{-1}{\ln b}}^{\,\to 0} + \frac{1}{\ln 1995} \right] = \frac{1}{\ln 1995}$$ converges.

$\therefore$ by the integral test the series converges

$\therefore$ the series is absolutely convergent

Question 2. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=2}^{\infty}\left[\ln(2n^2+1)-2\ln(n)\right] = \sum_{n=2}^{\infty}\left[\ln(2n^2+1)-\ln n^2\right]$$

$$= \sum_{n=2}^{\infty} \ln\left(\frac{2n^2+1}{n^2}\right)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2}\right) = \ln 2 \neq 0$$

$\therefore$ the series diverges by the n^{th} term divergence test.

Question 3. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$\sum_{n=3}^{\infty}\left[\left(\cos\left(2\pi-\frac{1}{n}\right)\right)^{n^2}\right]^{n}$ Let $a_n = \left[\left(\cos\left(2\pi-\frac{1}{n}\right)\right)^{n^2}\right]^{n}$

Lets apply the root test.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\left[\left[\cos\left(2\pi-\frac{1}{n}\right)\right]^{n^2}\right]^{n}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\cos\left(2\pi-\frac{1}{n}\right)\right)^{n^2} \quad \text{I.f. } 1^{\infty}$$

Let

$$y = \lim_{x\to\infty}\left(\cos\left(2\pi-\frac{1}{x}\right)\right)^{x^2}$$

$$\ln y = \ln \lim_{x\to\infty}\left(\cos\left(2\pi-\frac{1}{x}\right)\right)^{x^2}$$

$$\ln y = \lim_{x\to\infty} \ln\left(\cos\left(2\pi-\frac{1}{x}\right)\right)^{x^2}$$

$$\ln y = \lim_{x\to\infty} x^2\ln\left(\cos\left(2\pi-\frac{1}{x}\right)\right) \quad \text{I.f. } \infty\cdot 0$$

$$\ln y = \lim_{x\to\infty}\frac{\ln\left(\cos\left(2\pi-\frac{1}{x}\right)\right)}{\frac{1}{x^2}} \quad \text{I.f. } \frac{0}{0}$$

$$\ln y \overset{\hat{H}}{=} \lim_{x\to\infty}\frac{\frac{1}{\cos\left(2\pi-\frac{1}{x}\right)}\cdot -\sin\left(2\pi-\frac{1}{x}\right)\cdot\frac{+1}{x^2}}{\frac{-2}{x^3}}$$

$$\ln y = \lim_{x\to\infty}\frac{\tan\left(2\pi-\frac{1}{x}\right)}{\frac{2}{x}} \quad \text{I.f. } \frac{0}{0}$$

$$\ln y \overset{\hat{H}}{=} \lim_{x\to\infty}\frac{\sec^2\left(2\pi-\frac{1}{x}\right)\cdot\frac{1}{\cancel{x^2}}}{\frac{-2}{\cancel{x^2}}}$$

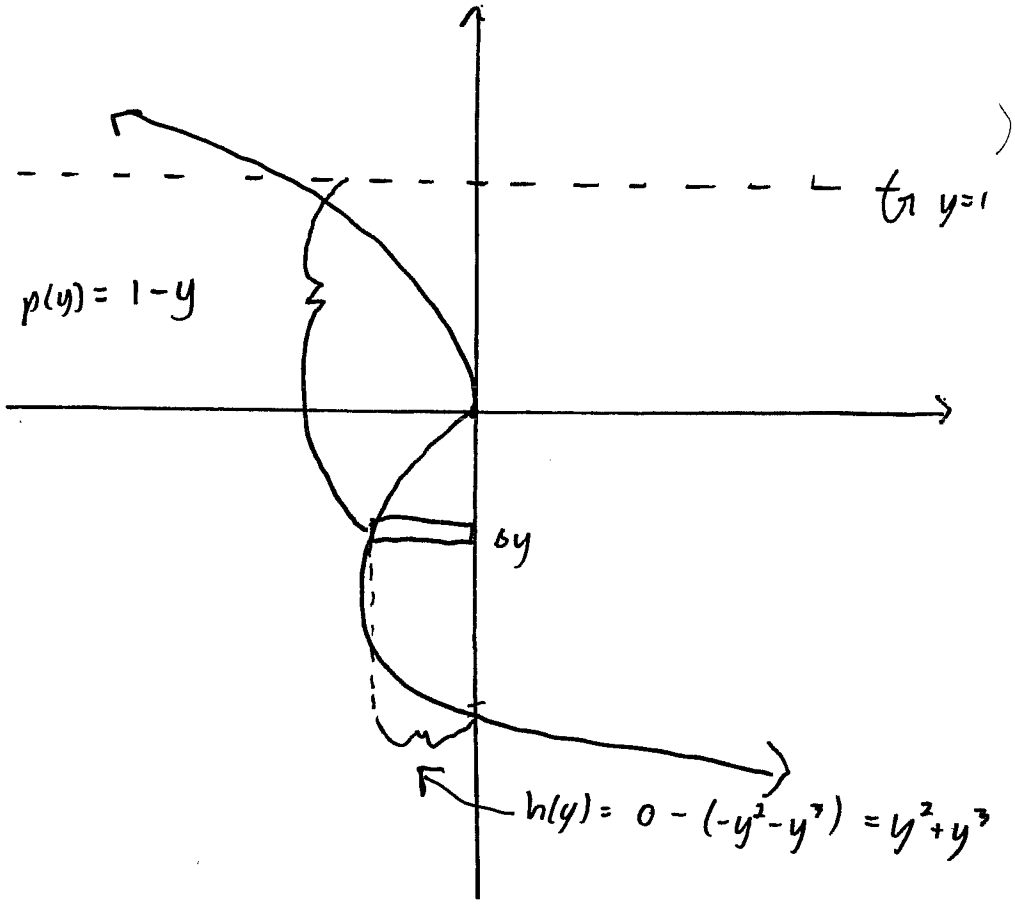
$$\ln y = \frac{-1}{2}$$

$$y = e^{-\frac{1}{2}}$$

So $\lim_{n\to\infty}\sqrt[n]{|a_n|} = e^{-\frac{1}{2}} < 1$

$\therefore$ converges by root test.

Question 4. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $x = -y^2 - y^3$, $x = 0$ rotated about the line $y = 1$. Sketch the region, draw a representative rectangle, label all parts of the graph and give the integral.



y-int: $0 = -y^2 - y^3$

$0 = -y^2(1+y)$

$y = 0 \quad y = -1$

$$V = \int_{-1}^{0} 2\pi\, p(y) h(y)\, dy$$

$$= \int_{-1}^{0} 2\pi (1-y)(y^2+y^3)\, dy$$

Question 5. *(5 marks)*

a. *(1 mark)* Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

$$\left\{\cos 1, \frac{\cos 2}{2}, \frac{\cos 3}{3}, \frac{\cos 4}{4}, \frac{\cos 5}{5}, \ldots\right\}$$

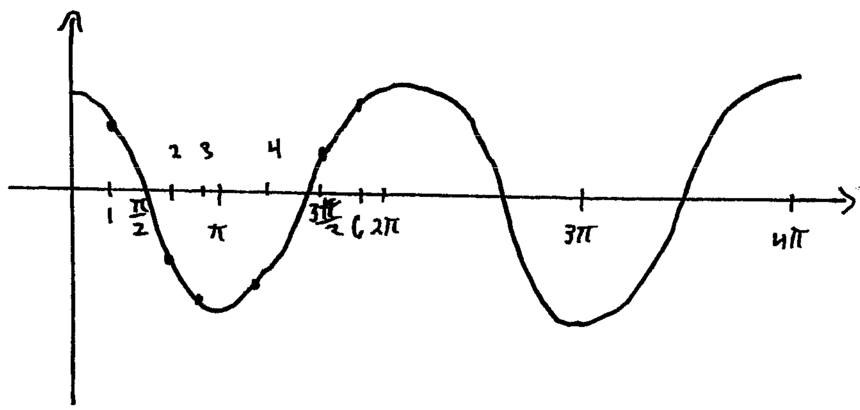
$a_n = \frac{\cos n}{n}$

b. *(1 mark)* Is a_n is monotonic?

c. *(1 mark)* Show that a_n is bounded.

d. *(2 marks)* Determine the limit of a_n as $n \to \infty$.

b)



not monotonic since $a_2 < a_1$ and $a_5 > a_4$

c) Since $-1 \le \cos n \le 1$ then $-1 \le \frac{\cos n}{n} \le 1$

$\therefore$ a_n is bounded by -1 and 1.

d) $b_n = \frac{-1}{n} \le \underset{\underset{\textstyle a_n}{\|}}{\frac{\cos n}{n}} \le \frac{1}{n} = c_n$

and $\lim\limits_{n \to \infty} b_n = \lim\limits_{n \to \infty} c_n = 0$

$\therefore$ by the squeeze thm. $a_n \to 0$ as $n \to \infty$

Question 6. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=2}^{\infty} \frac{2^n - 3^{n-1}}{4^{n+1}}$$

Lets look at both series independently

$$\sum_{n=2}^{\infty} \frac{2^n}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{2^n}{4 4^n} = \sum_{n=2} \frac{1}{4}\left(\frac{2}{4}\right)^n = \sum_{n=2}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^{n+2}$$

converges since geometric series where $r = \frac{1}{2} < 1$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{16}\left(\frac{1}{2}\right)^n$$

$$= \frac{1/16}{1-1/2} = \frac{1}{8}$$

$$\sum_{n=2}^{\infty} \frac{3^{n-1}}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{3^n}{12 4^n} = \sum_{n=2}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^n = \sum_{n=0}^{\infty} \frac{1}{12}\left(\frac{3}{4}\right)^{n+2}$$

converges since geometric series where $r = \frac{3}{4} < 1$

$$= \sum_{n=0}^{\infty} \frac{1}{12}\frac{3^2}{4^2}\left(\frac{3}{4}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{3}{64}\left(\frac{3}{4}\right)^n = \frac{3/64}{1-\frac{3}{4}}$$

$$= \frac{3}{16}$$

Since both series converges

$$\sum_{n=2}^{\infty} \frac{2^n - 3^{n-1}}{4^{n+1}} = \sum_{n=2}^{\infty} \frac{2^n}{4^{n+1}} - \sum_{n=2}^{\infty} \frac{3^{n-1}}{4^{n+1}}$$

$$= \frac{1}{8} - \frac{3}{16}$$

$$= \frac{-1}{16}$$

Question 7. *(5 marks)* Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\pi n - \arctan n}{\sqrt{n^5+n^2+1}}$$

Let $a_n = \dfrac{\pi n - \arctan n}{\sqrt{n^5+n^2+1}}$

$$a_n = \frac{\pi n - \arctan n}{\sqrt{n^5+n^2+1}} \leq \frac{\pi n}{\sqrt{n^5+n^2+1}} \leq \frac{\pi n}{\sqrt{n^5+n^2}} \leq \frac{\pi n}{\sqrt{n^5}} = \frac{\pi}{n^{3/2}} = b_n$$

$\sum b_n$ converges since p-series where $p = \frac{3}{2} > 1$

$\therefore$ by comparison test $\sum a_n$ converges.

Question 8. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=3}^{\infty}(-1)^n\sqrt{\frac{n^4+n}{n^5-n-3}}$$

Lets determine whether the series is absolutely convergent.

$$\sum_{n=3}^{\infty}\left|(-1)^n\sqrt{\frac{n^4+n}{n^5-n-3}}\right| = \sum_{n=3}^{\infty}\sqrt{\frac{n^4+n}{n^5-n-3}} = \sum_{n=3}^{\infty} a_n$$

$$a_n = \sqrt{\frac{n^4+n}{n^5-n-3}} \geq \sqrt{\frac{n^4}{n^5-n-3}} \geq \sqrt{\frac{n^4}{n^5-n}} \geq \sqrt{\frac{n^4}{n^5}} = \frac{1}{\sqrt{n}} = b_n$$

$\sum_{n=3}^{\infty} b_n$ diverges since p-series where $p=\frac{1}{2}<1$.

$\therefore \sum_{n=3}^{\infty} a_n$ diverges by comparison test.

$\therefore$ not absolutely convergent

Lets determine whether the series is conditionaly convergent.

Let $\sum_{n=3}^{\infty}(-1)^n b_n$ where $b_n = \sqrt{\frac{n^4+n}{n^5-n-3}}$ and $f(x)=\sqrt{\frac{x^4+x}{x^5-x-3}}$

① $b_{n+1} \leq b_n$

$$f'(x) = \frac{1}{2}\left(\frac{x^4+x}{x^5-x-3}\right)^{-\frac{1}{2}} \cdot \left[\frac{(4x^3+1)(x^5-x-3)-(5x^4-1)(x^4+x)}{(x^5-x-3)^2}\right]$$

$$= \frac{1}{2}\left(\frac{x^4+x}{x^5-x-3}\right)^{-\frac{1}{2}}\left[\frac{-x^8-4x^5-3x^4-3}{(x^5-x-3)^2}\right] < 0$$

$\therefore f(x)$ decreasing, so $f(n+1) < f(n)$
$b_{n+1} < b_n$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\sqrt{\frac{n^4+n}{n^5-n-3}} = 0$

$\therefore$ the series is convergent by the alternating series test.

$\therefore$ the series is conditionaly convergent.

Question 9. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent find its sum.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{3^n (n!)^2}$$

Let $a_n = \dfrac{(-1)^n (2n)!}{3^n (n!)^2}$

Lets apply the ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1} (2(n+1))!}{3^{n+1} ((n+1)!)^2}}{\dfrac{(-1)^n (2n)!}{3^n (n!)^2}} \right|$$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{\cancel{3^n} 3 (\cancel{n!}(n+1))^2} \cdot \frac{\cancel{3^n} \cancel{(n!)^2}}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{(2n)!} (2n+1)(2n+2)}{3 \cancel{(2n)!} (n+1)^2}$$

$$= \frac{4}{3} > 1$$

$\therefore$ diverges by the ratio test.

Bonus Question.

a. *(1 mark)* State the $K(\varepsilon)$ definition of the limit of a sequence.

b. *(3 marks)* Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$

c. *(2 marks)* If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1+a_n}$$

converges

~~[illegible] are bounded and monotonic~~ then ~~f~~ Find the limit of the sequence.

a) $\forall \varepsilon > 0 \ \exists K(\varepsilon) \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon \quad \forall n \geq K(\varepsilon)$

b) Given $\varepsilon > 0$ then $\exists K_1(\varepsilon)$ s.t. $|a_{2n} - L| < \varepsilon \quad \forall n \geq K_1(\varepsilon)$

then $\exists K_2(\varepsilon)$ s.t. $|a_{2n+1} - L| < \varepsilon \quad \forall n \geq K_2(\varepsilon)$

So let $K(\varepsilon) = \max\{2K_1(\varepsilon), 2K_2(\varepsilon)+1\}$ and it follows that if n is even or odd then

$$|a_n - L| < \varepsilon \quad \forall n \geq K(\varepsilon)$$

c) So $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} a_{n+1} = L$

$\therefore$ We have

$$a_{n+1} = 1 + \frac{1}{1+a_n}$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left[1 + \frac{1}{1+a_n}\right]$$

$$L = 1 + \frac{1}{1+L}$$

$$L - 1 = \frac{1}{1+L}$$

$$L^2 - 1 = 1$$

$$L^2 = 2$$

$$L = \pm\sqrt{2}$$

Since $a_n \geq 0$ then $L = \sqrt{2}$