Dawson College: Calculus I (CHEM. TECH.): 201-NYA-05-S04: Winter 2015

Name: Y. Lamontagne
Student ID: ______________

Test 1

This test is graded out of 43 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Evaluate the following limits, if possible. Justify.

a. *(2 marks)*

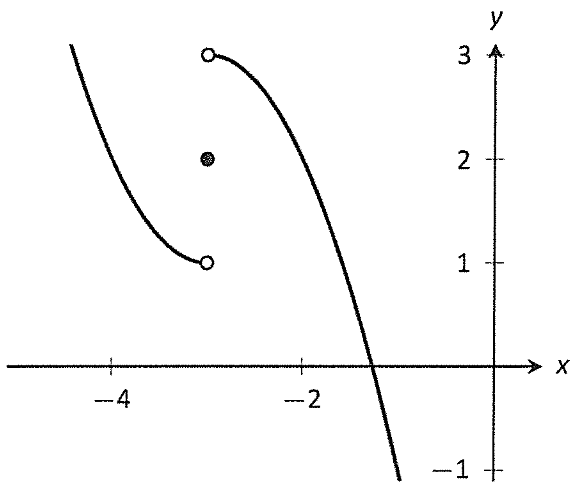
$$\lim_{x\to -5} \frac{x^2-x-30}{x+5} = \lim_{x\to -5} \frac{(x-6)\cancel{(x+5)}}{\cancel{(x+5)}} = \lim_{x\to -5} (x-6) = -5-6 = -11$$

b. *(2 marks)*

$$\lim_{x\to\infty} \frac{3x^3+2x^2+x}{99x^3+1} = \lim_{x\to\infty} \frac{3x^3+2x^2+x}{99x^3+1} \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{\frac{3x^3}{x^3}+\frac{2x^2}{x^3}+\frac{x}{x^3}}{\frac{99x^3}{x^3}+\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{3+\cancel{\frac{2}{x}}^{\to 0}+\cancel{\frac{1}{x^2}}^{\to 0}}{99+\cancel{\frac{1}{x^3}}^{\to 0}} = \frac{3}{99} = \frac{1}{33}$$

c. *(2 marks)* Determine the values of x for which the function, as represented below by the graph[1] doest not have a limit. Justify.



The limit does not exist at $x=-3$ since the limit from the left $\lim_{x\to 3^-} f(x) = 1$ is not equal to the limit from the right $\lim_{x\to 3^+} f(x) = 3$.

[1] from APEX Calculus

Question 2. *(5 marks)* Use the limit definition of the derivative to find the derivative of the function

$$f(x) = \frac{x+1}{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h+1}{x+h} - \frac{x+1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h+1}{x+h} \cdot \frac{x}{x} - \frac{x+1}{x} \cdot \frac{(x+h)}{x+h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x^2+xh+x}{x(x+h)} - \frac{x^2+xh+x+h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\cancel{x^2}+\cancel{xh}+\cancel{x}-\cancel{x^2}-\cancel{xh}-\cancel{x}-h}{x(x+h)}}{h}$$

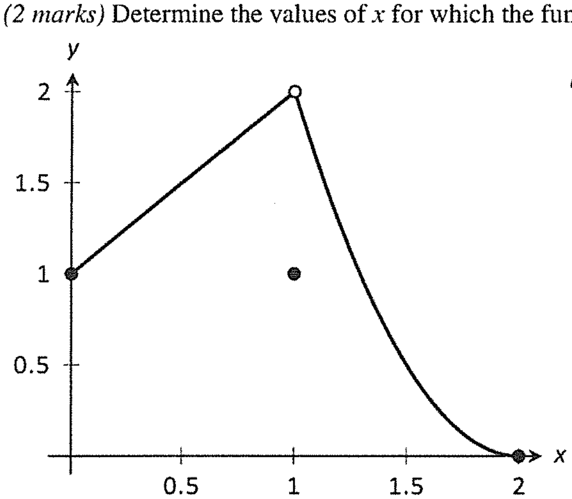
$$= \lim_{h \to 0} \frac{\frac{-h}{x(x+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\, x(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

Question 3.

a. *(2 marks)* State the conditions for a function, $f(x)$, to be continuous at $x = a$.

b. *(2 marks)* Determine the values of x for which the function, as represented below by the graph[2] is not continuous. Justify.



a) ① $f(a)$ is defined

② $\lim_{x \to a} f(x)$ exists

③ $\lim_{x \to a} f(x) = f(a)$

b) Not continuous at $x = 1$ since limit does not equal function at $x = 1$.

$f(1) = 1$

$\lim_{x \to 1} f(x) = 2 \neq f(1)$.

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[2]from APEX Calculus

Question 5. Let $f(x) = x^2 - x - 6$.

a. *(2 marks)* Find the equation of the tangent to the curve $f(x)$ at $x = 4$.

b. *(1 mark)* Find the equation of the normal to the curve $f(x)$ at $x = 2$.

c. *(3 marks)* Sketch the graph of $f(x)$ and the tangent from part a. and the normal from part b.

a) $f'(x) = 2x - 1$

$m_{tan} = f'(4) = 2(4) - 1 = 7$

$y = f(4) = 4^2 - 4 - 6 = 6$

So $(4, 6)$ is on tangent and curve

$y = m_{tan} x + b$

$6 = 7(4) + b$

$6 - 28 = b$

$b = -22$

$\therefore y = 7x - 22$

b) $m_{normal} = \frac{-1}{m_{tan}} = \frac{-1}{f'(2)} = \frac{-1}{2(2)-1} = -\frac{1}{3}$

$y = f(2) = 2^2 - 2 - 6$
$= -4$

So $(2, -4)$ is on normal and curve.

$y = m_{normal} x + b$

$y = -\frac{1}{3}x + b$

$-4 = -\frac{1}{3}(2) + b$

$b = -4 + \frac{2}{3} = \frac{-10}{3}$

$\therefore y = -\frac{1}{3}x - \frac{10}{3}$

c)

Sketch $f(x)$:

<u>x-int:</u> $0 = f(x)$

$0 = x^2 - x - 6$

$0 = (x-3)(x+2)$

$x = 3$ $x = -2$

<u>y-int:</u> $(0, -6)$

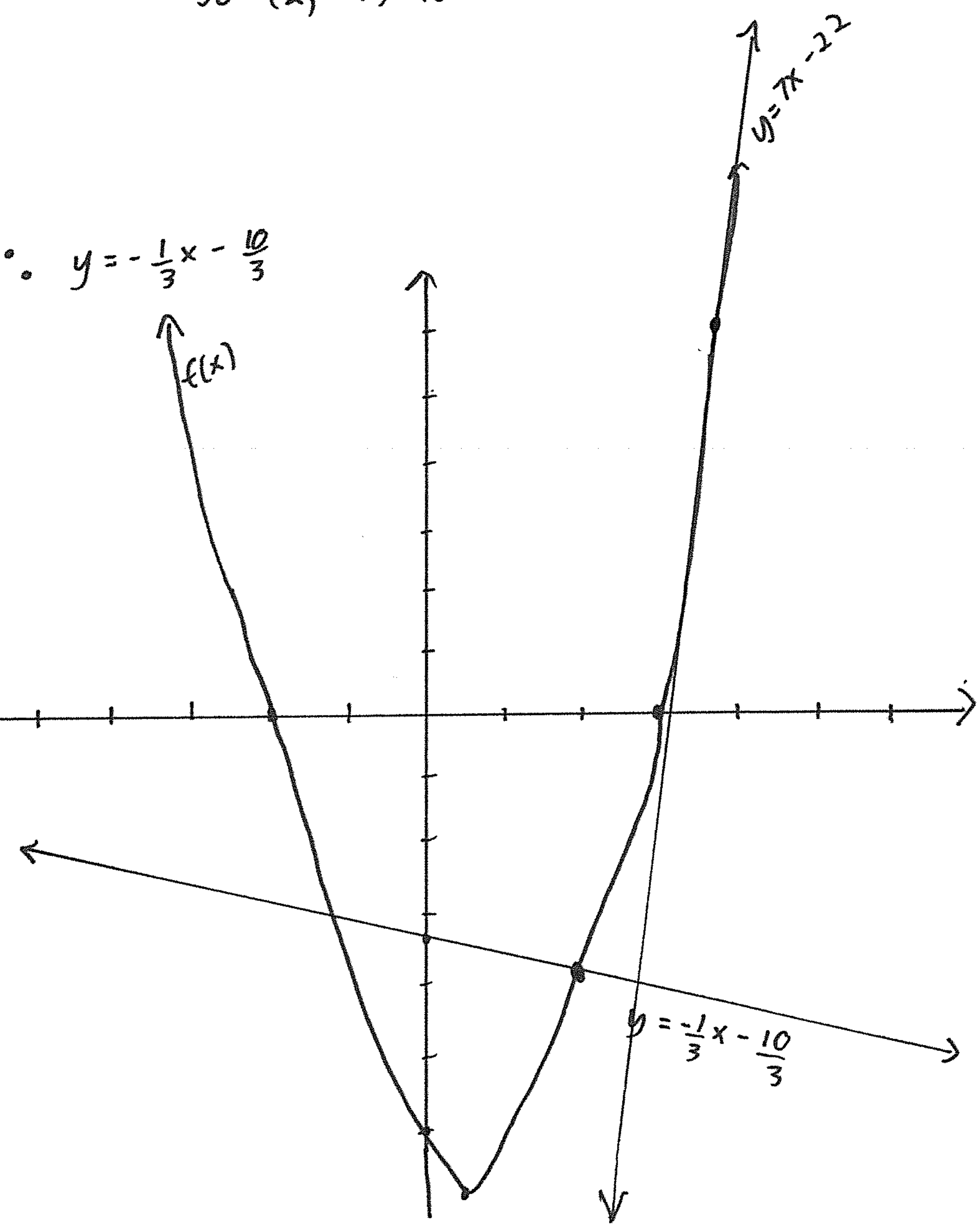
<u>vertex:</u> $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

$= \left(\frac{-(-1)}{2(1)}, f\left(\frac{-(-1)}{2(1)}\right)\right)$

$= \left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)$

$= \left(\frac{1}{2}, \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 6\right)$

$= \left(\frac{1}{2}, \frac{-25}{4}\right)$



Question 6. Find the derivative of the following functions:

a. *(3 marks)*

$$h(t) = \frac{t^2 + t^{3/2} + \sqrt{t} + 1}{t} = \frac{t^2}{t} + \frac{t^{3/2}}{t} + \frac{\sqrt{t}}{t} + \frac{1}{t} = t + t^{1/2} + t^{-1/2} + t^{-1}$$

$$h'(t) = 1 + \frac{1}{2}t^{-1/2} - \frac{1}{2}t^{-3/2} - t^{-2}$$

b. *(3 marks)*

$$f(z) = (z^3 + z^2)^2 (z^3 + z^2 + 1)^3$$

$$f'(z) = 2(z^3+z^2)(3z^2+2z)(z^3+z^2+1)^3 + (z^3+z^2)^2\, 3(z^3+z^2+1)^2(3z^2+2z)$$

$$= (z^3+z^2)(z^3+z^2+1)^2(3z^2+2z)\left[2(z^3+z^2+1) + 3(z^3+z^2)\right]$$

$$= (z^3+z^2)(z^3+z^2+1)^2(3z^2+2z)\left[5z^3+5z^2+2\right]$$

c. *(3 marks)*

$$y(x) = \sqrt{\frac{x^2+1}{x^3+1}}$$

$$= \left(\frac{x^2+1}{x^3+1}\right)^{1/2}$$

$$y'(x) = \frac{1}{2}\left(\frac{x^2+1}{x^3+1}\right)^{-\frac{1}{2}} \left[\frac{(2x)(x^3+1) - (x^2+1)(3x^2)}{(x^3+1)^2}\right]$$

$$= \frac{1}{2}\sqrt{\frac{x^3+1}{x^2+1}} \left[\frac{2x^4 + 2x - 3x^4 - 3x^2}{(x^3+1)^2}\right]$$

$$= \frac{1}{2}\sqrt{\frac{x^3+1}{x^2+1}} \left[\frac{-x^4 - 3x^2 + 2x}{(x^3+1)^2}\right]$$

Question 7. *(3 marks)* The Dawson chemistry students are experimenting with a new ultra fun compound called Mathemathium. While making some yields they discover that the concentration of the compound can be described by

$$C(t) = \frac{t^3+t+\pi}{t+e}$$

in mol/L and where t is time in seconds. What is the instantaneous rate of change of concentration after 2 minutes.

$$C'(t) = \frac{(3t^2+1)(t+e) - (t^3+t+\pi)}{(t+e)^2}$$

$$= \frac{3t^3+t+3et^2+e-t^3-t-\pi}{(t+e)^2}$$

$$= \frac{2t^3+3et^2-\pi+e}{(t+e)^2}$$

$$C'(120) = \frac{2(120)^3+3e(120)^2-\pi+e}{(120+e)^2}$$

$$= 237 \text{ mol/L/s}$$

Question 8. *(5 marks)* Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the relation

$$x^2 - xy = 1 - y^2$$

$$\frac{d}{dx}[x^2 - xy] = \frac{d}{dx}[1-y^2]$$

$$2x - y - xy' = -2yy'$$

$$2x - y = xy' - 2yy'$$

$$2x - y = y'(x-2y)$$

$$y' = \frac{2x-y}{x-2y}$$

$$\frac{d}{dx}[y'] = \frac{d}{dx}\left[\frac{2x-y}{x-2y}\right]$$

$$y'' = \frac{(2-y')(x-2y) - (2x-y)(1-2y')}{(x-2y)^2}$$

$$= \frac{\cancel{2x}-4y-y'x+2yy'-\cancel{2x}+4xy'+y-2yy'}{(x-2y)^2}$$

$$= \frac{-3y+3xy'}{(x-2y)^2}$$

$$= \frac{-3y+3x\left(\frac{2x-y}{x-2y}\right)}{(x-2y)^2}$$

$$= \frac{\frac{-3y(x-2y)+3x(2x-y)}{x-2y}}{(x-2y)^2}$$

$$= \frac{-3xy+6y^2+6x^2-3xy}{(x-2y)^3}$$

$$= \frac{6(y^2-xy+x^2)}{(x-2y)^3}$$

Bonus.

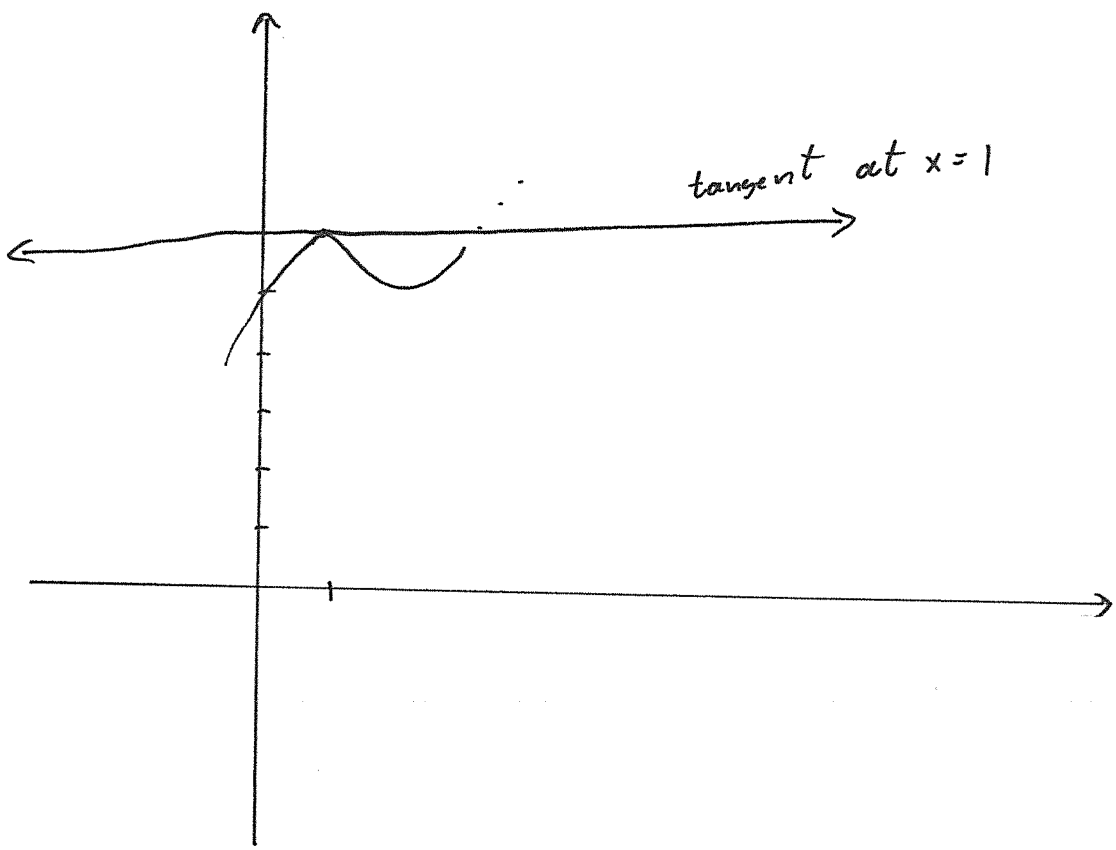
a. *(3 marks)* Why does Newton's Method fail in finding a root of $f(x) = 2x^3 - 9x^2 + 12x + 6$ when $x_0 = 1$? Give a geometrical justification.

b. *(2 marks)* Why does Newton's Method fail in finding a root of $f(x) = x^2 + x + 1$.

a) $f'(x) = 6x^2 - 18x + 12$

$f'(1) = 0$, so $x=1$ is a critical point so the tangent is horizontal. $\therefore$ will not intersect the x-axis.

So impossible to get x_1



b) The discriminant $\Delta = b^2 - 4ac$

$= 1^2 - 4(1)(1) = -3 < 0$

$\therefore$ no real roots

$\therefore$ Newton's Method can not find a root.