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Test 1

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.

a. *(6 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcrcrcl} 3x_1 & - & 2x_2 & + & x_3 & - & 3x_4 & + & x_5 & - & 4x_6 & = & 0 \\ 4x_1 & + & 3x_2 & - & x_3 & + & x_4 & - & 2x_5 & - & 3x_6 & = & 0 \\ 7x_1 & + & x_2 & & & - & 2x_4 & - & x_5 & + & 5x_6 & = & 0 \end{array}$$

b. *(1 mark)* Find a particular nontrivial solution to the above system.

c. *(1 mark)* Find a solution to the above system where $x_1 = 1$.

a) $\begin{bmatrix} 3 & -2 & 1 & -3 & 1 & -4 & 0 \\ 4 & 3 & -1 & 1 & -2 & -3 & 0 \\ 7 & 1 & 0 & -2 & -1 & 5 & 0 \end{bmatrix} \sim \begin{array}{l} \\ 3R_2 \to R_2 \\ 3R_3 \to R_3 \end{array} \begin{bmatrix} 3 & -2 & 1 & -3 & 1 & -4 & 0 \\ 12 & 9 & -3 & 3 & -6 & -9 & 0 \\ 21 & 3 & 0 & -6 & -3 & 15 & 0 \end{bmatrix}$

$\sim \begin{array}{l} \\ -4R_1 + R_2 \to R_2 \\ -7R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 3 & -2 & 1 & -3 & 1 & -4 & 0 \\ 0 & 17 & -7 & 15 & -10 & 7 & 0 \\ 0 & 17 & -7 & 15 & -10 & 43 & 0 \end{bmatrix} \sim \begin{array}{l} \\ \\ -R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 3 & -2 & 1 & -3 & 1 & -4 & 0 \\ 0 & 17 & -7 & 15 & -10 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 36 & 0 \end{bmatrix}$

$\sim \begin{array}{l} \frac{1}{3}R_1 \to R_1 \\ \frac{1}{17}R_2 \to R_2 \\ \frac{1}{36}R_3 \to R_3 \end{array} \begin{bmatrix} 1 & -2/3 & 1/3 & -1 & 1/3 & -4/3 & 0 \\ 0 & 1 & -7/17 & 15/17 & -10/17 & 7/17 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} 4/3 R_3 + R_1 \to R_1 \\ \sim -7/17 R_3 + R_2 \to R_2 \\ \\ \end{array} \begin{bmatrix} 1 & -2/3 & 1/3 & -1 & 1/3 & 0 & 0 \\ 0 & 1 & -7/17 & 15/17 & -10/17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\sim \begin{array}{l} 2/3 R_2 + R_1 \to R_1 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 1/17 & -7/17 & -1/17 & 0 & 0 \\ 0 & 1 & -7/17 & 15/17 & -10/17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ Let $\begin{array}{l} x_3 = s \\ x_4 = t \\ x_5 = u \end{array} \in \mathbb{R}$

$\therefore (x_1, x_2, x_3, x_4, x_5, x_6) = (-\frac{1}{17}s + \frac{7}{17}t + \frac{1}{17}u, \frac{7}{17}s - \frac{15}{17}t + \frac{10}{17}u, s, t, u, 0)$

c) if $s = t = 0$ and $u = 17$ then $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 10, 0, 0, 17, 0)$

b) $s = 1$ $t = u = 0$ then $(x_1, x_2, x_3, x_4, x_5, x_6) = (-\frac{1}{17}, \frac{7}{17}, 1, 0, 0, 0)$

Question 2. Consider the matrices:

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 \\ -3 & -4 & 0 \end{bmatrix} C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 3 & -1 \end{bmatrix} D = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} F = \begin{bmatrix} 5 & -1 \\ 0 & -2 \\ 1 & 0 \end{bmatrix}$$

a. *(2 marks)* Compute the following, if possible.

FAB

b. *(2 marks)* Compute the following, if possible.

$F^T AB^T$

c. *(2 marks)* Compute the following, if possible.

$(CD^{-1})^T$

d. *(5 marks)* Find E, if possible.

$(I - E^T)^{-1} = F^T AB^T - 12I$

$$FAB = \begin{bmatrix} 5 & -1 \\ 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -3 & -4 & 0 \end{bmatrix}$$

$3 \times \boxed{2 \neq 3} \times 3 \qquad 2\times 3$

not defined

$$F^T AB^T = \begin{bmatrix} 5 & 0 & 1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & -4 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 10 & 2 \\ -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & -4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 22 & -70 \\ -8 & 12 \end{bmatrix}$$

$$D^{-1} = \frac{1}{2(-2)-(1)(1)} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{-1}{5} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & -2/5 \end{bmatrix}$$

$$CD^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & -2/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2/5 & -4/5 \\ 1 & 1 \end{bmatrix}$$

$$(CD^{-1})^T = \begin{bmatrix} 1 & 2/5 & 1 \\ 0 & -4/5 & 1 \end{bmatrix}$$

d) $$(I - E^T)^{-1} = \begin{bmatrix} 22 & -70 \\ -8 & 12 \end{bmatrix} - \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$$

$$(I - E^T)^{-1} = \begin{bmatrix} 10 & -70 \\ -8 & 0 \end{bmatrix}$$

$$((I - E^T)^{-1})^{-1} = \frac{1}{-8(70)} \begin{bmatrix} 0 & 70 \\ 8 & 10 \end{bmatrix}$$

$$I - E^T = \begin{bmatrix} 0 & -1/8 \\ -1/70 & -1/56 \end{bmatrix}$$

$$E^T = I - \begin{bmatrix} 0 & -1/8 \\ -1/70 & -1/56 \end{bmatrix}$$

$$E^T = \begin{bmatrix} 1 & 1/8 \\ 1/70 & 57/56 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1/70 \\ 1/8 & 57/56 \end{bmatrix}$$

Question 3. Consider

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 7 \\ 3 & 2 & 0 \end{bmatrix}.$$

a. *(5 marks)* Find A^{-1}.

b. *(2 marks)* Determine $\mathrm{tr}(A^{-1})$ and $\mathrm{tr}(A^{-1}A)$.

c. *(3 marks)* Using a. solve $Ax = b$ where $x = [x_1, x_2, x_3]^T$ and $b = [-1, -2, 3]^T$.

a)

$$[A \mid I] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 7 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \begin{matrix} -2R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -4 & 7 & -2 & 1 & 0 \\ 0 & -4 & 0 & -3 & 0 & 1 \end{array}\right]$$

$$\sim R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 1 \\ 0 & -4 & 7 & -2 & 1 & 0 \end{array}\right] \sim \begin{matrix} \frac{1}{2}R_2 + R_1 \to R_1 \\ -\frac{1}{4}R_2 \to R_2 \\ -R_2 + R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 7 & 1 & 1 & -1 \end{array}\right]$$

$$\sim \frac{1}{7}R_3 \to R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} \end{array}\right] \quad \therefore A^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{4} & 0 & -\frac{1}{4} \\ \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} \end{bmatrix}$$

b)

$$\therefore \mathrm{tr}(A^{-1}) = -\frac{1}{2} + \frac{-1}{7} = \frac{-9}{14} \qquad \mathrm{tr}(A^{-1}A) = \mathrm{tr}(I) = 1+1+1 = 3$$

c) $Ax = b$

$A^{-1}Ax = A^{-1}b$

$Ix = A^{-1}b$

$$x = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{4} & 0 & -\frac{1}{4} \\ \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{3}{2} \\ -\frac{6}{7} \end{bmatrix}$$

Question 4. *(4 marks)* Let A be a square matrix. If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

Let $Ax=0$ and if it only has the trivial solution then A is invertible.

Let $x = By$ then $ABy = 0$

$Iy = 0$

$y=0$ So $x = B(0) = 0$

$\therefore A$ is invertible

$$AB = I$$
$$A^{-1}AB = A^{-1}I$$
$$IB = A^{-1}$$
$$B = A^{-1}$$

Question 5. *(4 marks)* Let $X = A^TA + B$ be a square matrix. Prove: If $B^2 = 0$, $AB = BA^T = 0$ then X^2 is symmetric.

$$X^2 = (A^TA+B)(A^TA+B)$$
$$= A^TAA^TA + A^TAB + BA^TA + B^2$$
$$= (A^TA)^2 + A^T0 + 0A + 0$$
$$= (A^TA)^2$$

Lets show $(X^2)^T = X^2$

$$\text{LHS} = (X^2)^T$$
$$= ((A^TA)^2)^T$$
$$= ((A^TA)^T)^2$$
$$= (A^T(A^T)^T)^2$$
$$= (A^TA)^2$$
$$= X^2$$

$\therefore X^2$ is symmetric.

Question 6. *(5 marks)* Show that

$$A = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

are row-equivalent by finding 3 elementary matrices E_i such that $E_3E_2E_1A = B$.

$$A \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\sim_{2R_3 \to R_3} \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \\ 8 & 10 & 12 \end{bmatrix}$$

$$\sim_{-R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 12 \end{bmatrix} = B$$

$$I \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$I \sim_{2R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = E_2$$

$$I \sim_{-R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

Question 7. The augmented matrix of a linear system is given by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \pi \\ 0 & \sqrt{2} & 4 & 5 & 6 \\ 0 & 0 & 0 & a & b \end{bmatrix}$$

If possible for what values of a and b there is

a. *(2 marks)* no solution? Justify.

b. *(2 marks)* exactly one solution? Justify.

c. *(2 marks)* infinitely many solutions? Justify.

a) If $a=0$ and $b \neq 0$ then the last row of the matrix as an equation $0x_1 + 0x_2 + 0x_3 + ax_4 = b$

$0x_4 = b \neq 0$

no value for x_4 can satisfy the above.

b) Impossible, since # leading 1's < # var.

c) $a, b \in \mathbb{R}$ except when $a=0$ and $b \neq 0$. Since no matter what a, b are and there is a solution, x_3 is a free variable. Hence infinite amount of solution

Bonus Question.[1] *(5 marks)*

"This system of n linear equations with n unknowns," said the Great Mathematician, "as a curious property."

"Good heavens!" said the administrator, "What is it?" "Note," said the Great Mathematician, "that the constants are in arithmetic progression."

"It's all so clear when you explain it!" said the administrator. "Do you mean like $6x+9y=12$ and $15x+18y=21$?"

"Quite so," said the Great Mathematician, pulling out his bassoon. "Indeed, the system has a unique solution. Can you find it?"

"Good heavens!" cried the administrator, "I am baffled"

Are you?

In order to be a system $n > 1$

Let $n = 2$ then

$(a+d)x + (2a+d)y = 3a+d$
$(4a+d)x + (5a+d)y = 6a+d$

then $x=-1$ and $y=2$ is a solution to the system. And since eqn 1 is not a multiple of eqn 2. The solution is unique.

[1] modified from Underwood Dudley, Arnold Lebow (proposers), David Rothman (solver), Elementary problem 1151, American Mathematical Monthly, vol. 70 no. 1 (Jan. 1963), p. 93.

Let $n > 3$

(1) $(a+d)x_1 + (2a+d)x_2 + (3a+d)x_3 + \dots + (na+d)x_n = (n+1)a+d$

(2) $((n+2)a+d)x_1 + ((n+3)a+d)x_2 + ((n+4)a+d)x_3 + \dots + ((2n+1)a+d)x_n = (2n+2)a+d$

(3) $((2n+3)a+d)x_1 + ((2n+4)a+d)x_2 + ((2n+5)a+d)x_3 + \dots + ((3n+2)a+d)x_n = (3n+3)a+d$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$

(n) $(n^2a+d)x_1 + ((n^2+1)a+d)x_2 + ((n^2+2)a+d)x_3 + \dots + ((n^2+(n-1))a+d)x_n = (n^2+n)a+d$

If the above is put in an augmented matrix and the following elementary row operations are performed

$$\begin{matrix} -R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{matrix} \left[\quad \right] \sim -2R_2+R_3 \to R_3 \left[\begin{array}{ccccc|c} x & x & x & & x & x \\ x & x & x & & x & x \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x & x & x & & x & x \end{array}\right]$$

$\therefore$ # leading 1 < # vars

So infinite amount of solutions

$\therefore$ unique solution only when $n = 2$