Dawson College: **Linear Algebra**: 201-NYC-05-S03: Winter 2015

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Test 2

This test is graded out of ~~40~~ 38 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$\begin{array}{llcll} \mathscr{A}: & (1,-2,3) & & & \\ \mathscr{L}_1: & (x,y,z) & = & (1+t,3+2t,5-t) & t\in\mathbb{R} \\ \mathscr{L}_2: & (x,y,z) & = & (2+2s,4,6-3s) & s\in\mathbb{R} \\ \mathscr{P}_1: & x-2y-3z-\pi & = & 0 & \end{array}$$

a. *(1 mark)* Determine the equation of the line $\mathscr{L}_3$ that passes through $\mathscr{A}$ and is perpendicular to $\mathscr{P}_1$.

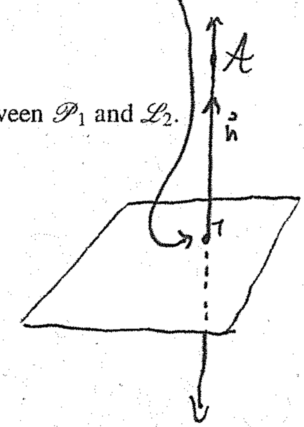
b. *(2 marks)* Determine the point on $\mathscr{P}_1$ that is closest to $\mathscr{A}$.

c. *(3 marks)* Determine the point on $\mathscr{L}_1$ that is closest to $\mathscr{A}$.

d. *(2 marks)* Are $\mathscr{P}_1$ and $\mathscr{L}_1$ parallel, perpendicular, or neither, justify?

e. *(3 marks)* Are $\mathscr{P}_1$ and $\mathscr{L}_2$ parallel, perpendicular, or neither, justify? Find the shortest distance between $\mathscr{P}_1$ and $\mathscr{L}_2$.

point of intersection
closest point to
$\mathscr{A}$



a) $\mathscr{L}_3: (x,y,z) = \mathscr{A} + t\vec{n} \quad t\in\mathbb{R}$ where $\vec{n} = (1,-2,-3)$

b) Sub $\mathscr{L}_3$ into $\mathscr{P}_1$, $\mathscr{L}_3: (x,y,z) = (1+t, -2-2t, +3-3t)$

So $(1+t) -2(-2-2t) -3(3-3t) - \pi = 0$

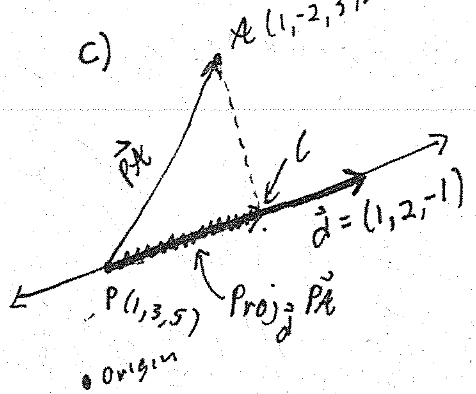
$1+t+4+4t-9+9t - \pi = 0$

$14t = \pi + 4$

$t = \frac{\pi+4}{14}$

$\therefore$ closest point is $(x,y,z) = (1,-2,3) + \frac{\pi+4}{14}(1,-2,-3) = \left(1+\frac{\pi+4}{14}, -2-\frac{2(\pi+4)}{14}, 3-\frac{3(\pi+4)}{14}\right)$

c)



$\vec{PA} = A - P$
$= (1,-2,3)-(1,3,5)$
$= (0,-5,-2)$

$\vec{OC} = \vec{OP} + \text{proj}_{\vec{d}}\vec{PA}$

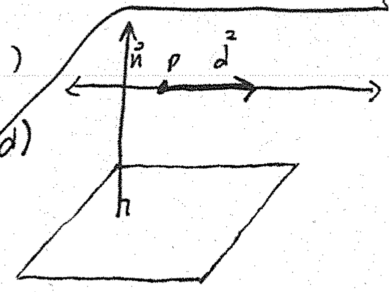
$= (1,3,5) + \frac{(0,-5,-2)\cdot(1,2,-1)}{(1,2,-1)\cdot(1,2,-1)}(1,2,-1)$

$= (1,3,5) + \frac{-8}{1^2+2^2+(-1)^2}(1,2,-1)$

$= (1,3,5) - \frac{4}{3}(1,2,-1)$

$= \left(-\frac{1}{3}, \frac{1}{3}, \frac{19}{3}\right)$

d)



The plane and line are parallel if $\vec{n}\cdot\vec{d} = 0$, so

$\vec{n}\cdot\vec{d} = (1,-2,-3)\cdot(1,2,-1)$

$= 1-4+3 = 0 \quad \therefore$ parallel

e) Since $\vec{n}\cdot\vec{d} = (1,-2,-3)\cdot(2,0,-3) = 2+9 = 11 \neq 0 \quad \therefore$ not parallel

Since $\vec{n} \neq k\vec{d} \quad \therefore$ not perpendicular

$\therefore$ the line and plane must intersect $\therefore$ the distance is 0.

Question 2. Given

$$A=\begin{bmatrix}10&1&1&1&9&3&4&1&0&9\\0&9&1&1&4&9&2&7&7&9\\0&0&8&1&1&4&9&2&7&7\\0&0&0&7&1&1&4&9&2&7\\0&0&0&0&6&1&1&4&9&2\\0&0&0&0&0&5&1&1&4&9\\0&0&0&0&0&0&4&1&1&4\\0&0&0&0&0&0&0&3&1&1\\0&0&0&0&0&0&0&0&2&1\\0&0&0&0&0&0&0&0&0&1\end{bmatrix},\quad B=\begin{bmatrix}1&1&2&6\\2&3&3&-3\\-3&3&1&9\\0&1&-1&-12\end{bmatrix},\quad C=\begin{bmatrix}-1&3&2&3\\2&0&3&0\\-4&0&1&2\\1&5&0&-3\end{bmatrix}$$

a. *(4 marks)* Evaluate $\det(2\,\mathrm{adj}(A)+3A^{-1})$, if possible.

b. *(4 marks)* If D is a 4×4 matrix and $B^{-1}D^2=I$ then determine $\det(D)$, if possible.

c. *(4 marks)* Compute $\det(C)$.

a) $\det(A) = 10\cdot 9\cdot 8\cdot 7\cdot\ldots\cdot 3\cdot 2\cdot 1 = 10!$ (since A is triangular)

and $\det(\mathrm{adj}(A)) = (\det(A))^{n-1} = (10!)^9$

and $A^{-1} = \dfrac{1}{\det A}\,\mathrm{adj}\,A$. So $\det\left(2\,\mathrm{adj}(A) + \dfrac{3}{\det A}\,\mathrm{adj}(A)\right)$

$= \det\left(\dfrac{2\det A + 3}{\det A}\,\mathrm{adj}\,A\right)$

$= \left(\dfrac{2\det A + 3}{\det A}\right)^{10}\det(\mathrm{adj}\,A) = \left(\dfrac{2\cdot 10! + 3}{10!}\right)^{10}(10!)^9$

$= \dfrac{(2\cdot 10! + 3)^{10}}{10!}$

b) $\det(B^{-1}D^2) = \det(I)$

$\det(B^{-1})\det(D^2) = 1$

$\dfrac{1}{\det B}(\det D)^2 = 1$

$(\det D)^2 = \det B$

$$B = \begin{bmatrix}1&1&2&6\\2&3&3&-3\\-3&3&1&9\\0&1&-1&-12\end{bmatrix} \sim \begin{matrix}-2R_1+R_2\to R_2\\3R_1+R_3\to R_3\end{matrix}\begin{bmatrix}1&1&2&6\\0&1&-1&-15\\0&6&7&27\\0&1&-1&-12\end{bmatrix} \sim \begin{matrix}-6R_2+R_3\to R_3\\-R_2+R_4\to R_4\end{matrix}\begin{bmatrix}1&1&2&6\\0&1&-1&-15\\0&0&13&117\\0&0&0&3\end{bmatrix} = E$$

$\det(B) = \det(E)$ So $(\det D)^2 = 3\cdot 13$

$\det D = \pm\sqrt{39}$.

c)

$\det(C) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$

$$= 3(-1)^{2+1}\begin{vmatrix}2&3&0\\-4&1&2\\1&0&-3\end{vmatrix} + 0 + 0 + 5(-1)^{2+4}\begin{vmatrix}-1&2&3\\2&3&0\\-4&1&2\end{vmatrix}\begin{matrix}-1&2\\2&3\\-4&1\end{matrix}$$

$= -3\left[a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}\right] + 5\left[(-1)(3)(2) + 2(0)(-4) + 3(2)(1) - (3)(3)(-4) - (-1)(0)(1) - (2)(2)(2)\right]$

$$= -3\left[2\begin{vmatrix}1&2\\0&-3\end{vmatrix} - 3\begin{vmatrix}-4&2\\1&-3\end{vmatrix} + 0\right] + 5\,[28]$$

$= -3\left[2(-3) - 3(12-2)\right] + 5(28) = 248$

Question 3.[1] *(2 marks)* Prove or disprove: If N is a matrix such that $\det(N) = 1$ then $N^2 = I$. Disprove

$N = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $N^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

$\det(N) = 2(1) - (1)(1)$
$= 1$

Question 4. *(3 marks)* Consider

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & x_3 & = & 1 \\ & & \alpha x_2 & - & \beta x_3 & = & 1 \\ & & 2x_2 & - & x_3 & = & -1 \end{array}$$

Let $A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & \alpha & -\beta \\ 0 & 2 & -1 \end{bmatrix}$

where the determinant of the coefficient matrix is 1 and the x_2 component of the solution is 1. Determine α and β, if possible. *(hint: use Cramer's rule)*

$|A| = 3\begin{vmatrix} \alpha & -\beta \\ 2 & -1 \end{vmatrix} = 3[-\alpha + 2\beta] = 1$

$A_2 = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & -\beta \\ 0 & -1 & -1 \end{bmatrix}$ $|A_2| = 3[-1-\beta]$

$X_2 = \dfrac{|A_2|}{|A|} = 1$

$\dfrac{-3-3\beta}{-3\alpha+6\beta} = 1$

$-3-3\beta = -3\alpha + 6\beta$
$-3 = -3\alpha + 9\beta$
$1 = \alpha - 3\beta$ ①

and

$1 = -3\alpha + 6\beta$ ②

$3① + 2$

$4 = -3\beta$

$\beta = \frac{-4}{3}$

So $\alpha = -3$

not unique solution.

Question 5. §3.3 # 21

a. *(2 marks)* The equation $3x + 2y + z = 6$ can be viewed as a linear system of one equation in three unknowns. Express a general solution of this equation as a particular solution plus a general solution of the associated homogeneous system.

b. *(1 mark)* Give a geometric interpretation of the result in part a.

a) a particular solution of the system is $X_0 = (2,0,0)$
The general solution of the homogeneous system is: $3x+2y+z=0$

Let $y=s$
$z=t$
$x = -\frac{2}{3}s - \frac{1}{3}t$

$\therefore (x,y,z) = s(-\frac{2}{3}, 1, 0) + t(-\frac{1}{3}, 0, 1)$

$\therefore$ the general solution is: $(x,y,z) = (2,0,0) + s(-\frac{2}{3},1,0) + t(-\frac{1}{3},0,1)$

b) The solution set is a plane passing through $(2,0,0)$.

[1]Modified from a John Abbott final examination problem

Question 7.[2] A triangle is created by joining the $x-$, $y-$, and $z-$intercepts of the plane $x + 2y + 3z = 12$.

a. *(1 mark)* Sketch the plane.

b. *(1 mark)* Find the coordinates of the vertices of the triangle.

c. *(2 marks)* Find the area of the triangle.

d. *(3 marks)* If θ is the angle between the plane $x + 2y + 3z = 12$ and the xy-plane, find the value of $\cos\theta$.

a) x-int: let $y=z=0$ $x=12$ $(12,0,0)$

b) y-int: let $x=z=0$ $y=6$ $(0,6,0)$

z-int: let $x=y=0$ $z=4$ $(0,0,4)$

c) $\text{area}\,\Delta = \dfrac{(\text{base})(\text{height})}{2}$

$= \dfrac{\|\vec{AC}\|\|\vec{w}_2\|}{2} = \dfrac{\sqrt{180}\,\frac{\sqrt{1120}}{5}}{2} \doteq 12\sqrt{14}$

$\vec{AC} = C - A = (0,6,0)-(12,0,0) = (-12,6,0)$

$\vec{AB} = B - A = (0,0,4)-(12,0,0) = (-12,0,4)$

$\|\vec{AC}\| = \sqrt{(-12)^2+6^2+0^2} = \sqrt{180}$

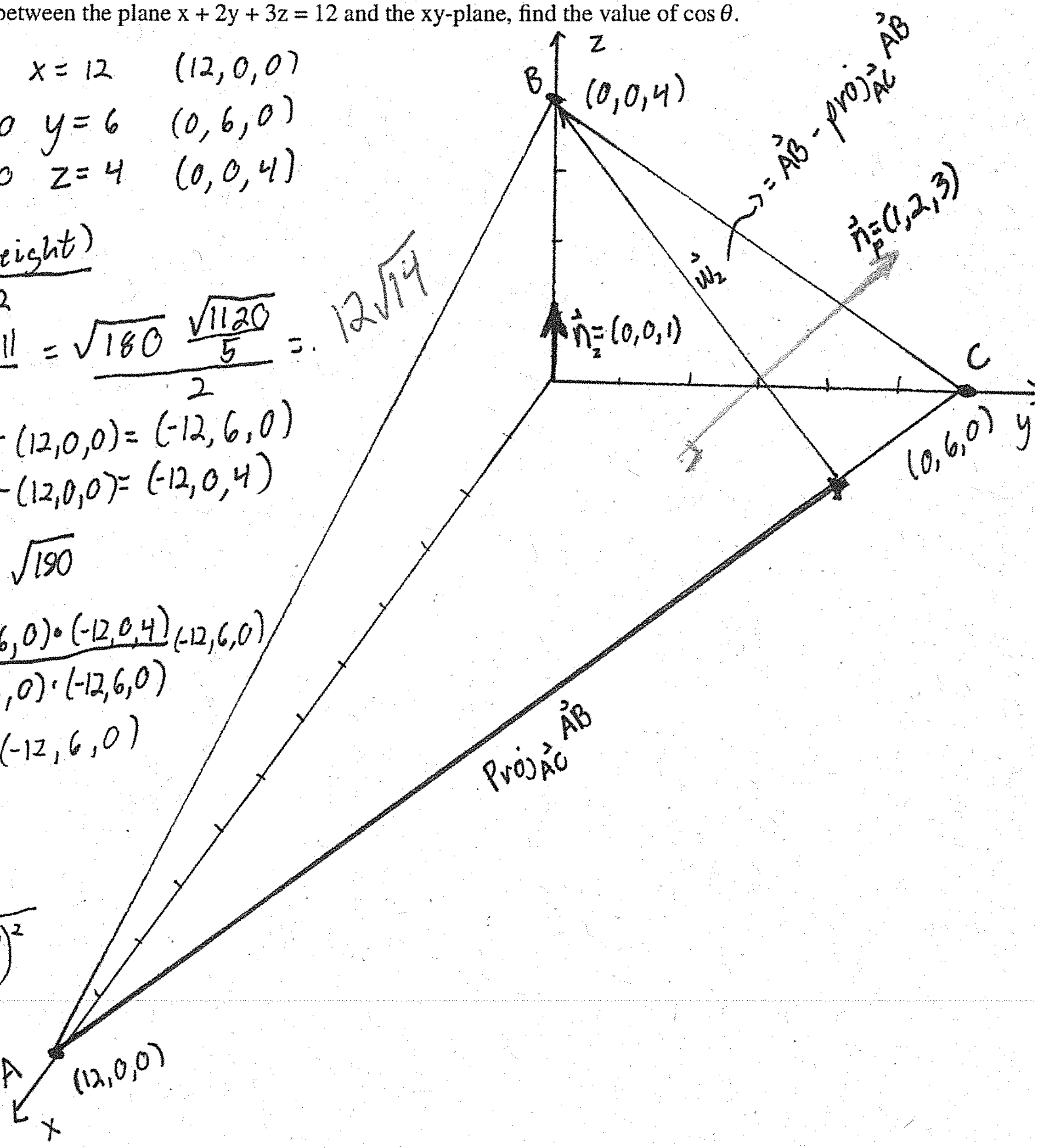
$\vec{w}_2 = (-12,0,4) - \dfrac{(-12,6,0)\cdot(-12,0,4)}{(-12,6,0)\cdot(-12,6,0)}(-12,6,0)$

$= (-12,0,4) - \dfrac{144}{180}(-12,6,0)$

$= \left(\dfrac{-12}{5}, \dfrac{-24}{5}, 4\right)$

$\|\vec{w}_2\| = \sqrt{\left(\frac{-12}{5}\right)^2+\left(\frac{-24}{5}\right)^2+\left(\frac{20}{5}\right)^2}$

$= \dfrac{\sqrt{1120}}{5}$



d) $\vec{n}_z \cdot \vec{n}_p = \|\vec{n}_z\|\|\vec{n}_p\|\cos\theta$

$(0,0,1)\cdot(1,2,3) = \|(0,0,1)\|\|(1,2,3)\|\cos\theta$

$3 = 1\cdot\sqrt{1^2+2^2+3^2}\cos\theta$

$3 = \sqrt{14}\cos\theta$

$\cos\theta = \dfrac{3}{\sqrt{14}}$

$\theta = \arccos\left(\dfrac{3}{\sqrt{14}}\right) \doteq 36.7$

So angle $90^\circ - 36.7^\circ$
$= 53.3^\circ$

[2]Modified from a John Abbott final examination problem

Bonus Question. *(5 marks)* CAUCHY-SCHWARTZ INEQUALITY: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u} \cdot \vec{v}| \leq ||\vec{u}||||\vec{v}||$. Prove the CAUCHY-SCHWARTZ INEQUALITY without assuming that the law of cosine holds in $\mathbb{R}^n$. *(hint: look at the squared norm of $||\vec{u}||\vec{v} - ||\vec{v}||\vec{u}$ and $||\vec{u}||\vec{v} + ||\vec{v}||\vec{u}$)*

$$0 \leq \| \|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}\|^2 = (\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u}) \cdot (\|\vec{u}\|\vec{v} - \|\vec{v}\|\vec{u})$$
$$= \|\vec{u}\|\|\vec{u}\|\vec{v}\cdot\vec{v} - \|\vec{u}\|\|\vec{v}\|\vec{u}\cdot\vec{v} - \|\vec{v}\|\|\vec{u}\|\vec{u}\cdot\vec{v} + \|\vec{v}\|\|\vec{v}\|\vec{u}\cdot\vec{u}$$
$$= \|\vec{u}\|^2\|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\vec{u}\cdot\vec{v} + \|\vec{v}\|^2\|\vec{u}\|^2$$
$$= 2\|\vec{u}\|^2\|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\vec{u}\cdot\vec{v}$$

$$2\|\vec{u}\|\|\vec{v}\|\vec{u}\cdot\vec{v} \leq 2\|\vec{u}\|^2\|\vec{v}\|^2$$
$$\vec{u}\cdot\vec{v} \leq \|\vec{u}\|\|\vec{v}\|$$

$$0 \leq \| \|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u}\|^2 = (\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u}) \cdot (\|\vec{u}\|\vec{v} + \|\vec{v}\|\vec{u})$$
$$= \|\vec{u}\|\|\vec{u}\|\vec{v}\cdot\vec{v} + \|\vec{u}\|\|\vec{v}\|\vec{v}\cdot\vec{u} + \|\vec{v}\|\|\vec{u}\|\vec{u}\cdot\vec{v} + \|\vec{v}\|\|\vec{v}\|\vec{u}\cdot\vec{u}$$
$$= \|\vec{u}\|^2\|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\|\vec{u}\cdot\vec{v} + \|\vec{v}\|^2\|\vec{u}\|^2$$
$$= 2\|\vec{u}\|^2\|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\|\vec{u}\cdot\vec{v}$$

$$-2\|\vec{u}\|\|\vec{v}\|\vec{u}\cdot\vec{v} \leq 2\|\vec{u}\|^2\|\vec{v}\|^2$$
$$-\vec{u}\cdot\vec{v} \leq \|\vec{u}\|\|\vec{v}\|$$

$$\therefore |\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$$