Dawson College: **Linear Algebra**: 201-NYC-05-S03: Winter 2015

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Test 3

This test is graded out of 38 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$\begin{array}{llll} \mathscr{L}_1: & (x,y,z) = & (1+t, 3+2t, 5-t) & t \in \mathbb{R} \\ \mathscr{L}_2: & (x,y,z) = & (2+2s, 4, 6-3s) & s \in \mathbb{R} \end{array}$$

a. *(5 marks)* Are $\mathscr{L}_1$ and $\mathscr{L}_2$ parallel, perpendicular, or neither, justify? Find the points on $\mathscr{L}_1$ and $\mathscr{L}_2$ which are closest to each other.

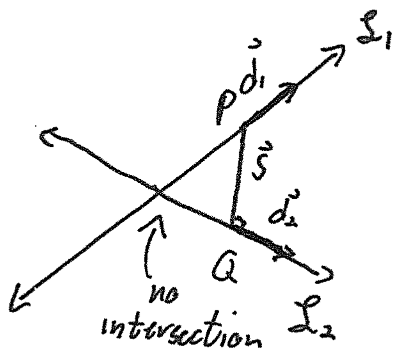
b. *(2 marks)* Find the shortest distance between $\mathscr{L}_1$ and $\mathscr{L}_2$.

a) $\mathscr{L}_1$ and $\mathscr{L}_2$ are not parallel since $\vec{d}_1 = (1,2,-1) \neq k(2,0,-3) = k\vec{d}_2$

$\mathscr{L}_1$ and $\mathscr{L}_2$ are not perpendicular since $\vec{d}_1 \cdot \vec{d}_2 = 1(2)+2(0)+(-1)(-3) = 5 \neq 0$

$\therefore$ $\mathscr{L}_1$ and $\mathscr{L}_2$ are not parallel nor perpendicular

The lines are possibly skew.



$\vec{S} = (2+2s, 4, 6-3s) - (1+t, 3+2t, 5-t)$
$= (1+2s-t, 1-2t, 1-3s+t)$

$\vec{d}_1 \cdot \vec{S} = 0$
$0 = 1+2s-t + 2(1-2t) - (1-3s+t)$
$0 = 5s - 6t + 2 \quad \Rightarrow \quad s = \dfrac{-2+6t}{5}$ (1)

$\vec{d}_2 \cdot \vec{S} = 0$
$0 = 2(2s-t+1) - 3(1-3s+t)$
$0 = -1 - 5t + 13s$ (2)

sub (1) into (2)

$-1 - 5t + 13\left(\dfrac{-2+6t}{5}\right) = 0$

$t = \dfrac{31}{53}$ sub into (1)

$s = \dfrac{-2 + 6\left(\frac{31}{53}\right)}{5}$

$= \dfrac{16}{53}$

So $P = \left(1+\frac{31}{53}, 3+2\left(\frac{31}{53}\right), 5-\frac{31}{53}\right)$
$= \left(\frac{84}{53}, \frac{221}{53}, \frac{234}{53}\right)$

$Q = \left(2+2\left(\frac{16}{53}\right), 4, 6-3\left(\frac{16}{53}\right)\right)$
$= \left(\frac{138}{53}, 4, \frac{270}{53}\right)$

b) $\vec{S} = \left(1+2\left(\frac{16}{53}\right) - \frac{31}{53}, 1-2\left(\frac{31}{53}\right), 1-3\left(\frac{16}{53}\right)+\frac{31}{53}\right)$

$= \left(\frac{54}{53}, \frac{-9}{53}, \frac{36}{53}\right)$

distance $= \dfrac{1}{53}\sqrt{54^2+(-9)^2+36^2} = \dfrac{\sqrt{4293}}{53}$

Question 2.[1] Let $\mathscr{H} = \{X \mid X \in \mathscr{M}_{2\times 2} \text{ and } AX - X = 0\}$ where $A = \begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix}$ and with the usual addition and scalar multiplication.

a. *(2 marks)* Give an example of a non-zero matrix in $\mathscr{H}$. Justify.

b. *(2 marks)* Does $\mathscr{H}$ satisfy closure under vector addition? Justify.

c. *(2 marks)* Does $\mathscr{H}$ contain the zero vector of $\mathscr{M}_{2\times 2}$ (the vector space of 2×2 matrices)? Justify.

d. *(2 marks)* Does $\mathscr{H}$ satisfy closure under scalar multiplication? Justify.

e. *(2 marks)* Is $\mathscr{H}$ a vector subspace of $\mathscr{M}_{2\times 2}$ (the vector space of 2×2 matrices)? Justify.

a) Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2x2}$

$$AX = X$$

$$\begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2a-2c & 2b-2d \\ 2a-3c & 2b-3d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{cases} 2a-2c = a \\ 2a-3c = c \end{cases} \qquad \begin{cases} 2b-2d = b \\ 2b-3d = d \end{cases}$$

$$a - 2c = 0 \qquad b-2d = 0$$
$$2a-4c = 0 \qquad 2b-4d = 0 \Big\} \to b = 2d$$

$\therefore a = 2c$

$\therefore X = \begin{bmatrix} 2c & 2d \\ c & d \end{bmatrix} \in H$ So $X = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \in H$

b) Let $X, Y \in H$ then $X + Y \in H$ since $A(X+Y) - (X+Y)$

$$= AX + AY - X - Y$$
$$= \underbrace{AX - X} + \underbrace{AY - Y}$$
$$= 0 + 0 \quad \text{since } X, Y \in H$$
$$= 0$$

c) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$ since $A0 - 0$

$$= 0 - 0$$
$$= 0$$

d) Let $X \in H$ and $r \in \mathbb{R}$ then $rX \in H$ since $A(rX) - rX$

$$= rAX - rX$$
$$= r(AX - X)$$
$$= r0 = 0$$

e) H is a subspace of M_{2x2} since it is closed under addition and scalar multiplication.

[1] From a John Abbott final examination

Question 3. *(2 marks)* Determine whether the following is a vector space:

$$\mathscr{Y} = \{(1,y) \mid y \in \mathbb{R}\}$$

with the following addition and scalar multiplication.

$$(1,y_1)+(1,y_2) = (1, y_1^2 + y_2^2)$$

and

$$k(1,y) = (1,ky)$$

If it is not a vector space, clearly state and show which axiom fails.

The above is not a vector space since it is not associative

Let $(1,y_1),(1,y_2),(1,y_3) \in \mathscr{Y}$

$[(1,y_1)+(1,y_2)]+(1,y_3)$
$= (1, y_1^2+y_2^2) + (1,y_3)$
$= (1, (y_1^2+y_2^2)^2 + y_3^2)$
$= (1, y_1^4 + 2y_1^2y_2^2 + y_2^4 + y_3^2)$ $\neq$

$(1,y_1) + [(1,y_2)+(1,y_3)]$
$= (1,y_1) + (1, y_2^2+y_3^2)$
$= (1, y_1^2 + (y_2^2+y_3^2))$
$= (1, y_1^2 + y_2^4 + 2y_2^2y_3^2 + y_3^4)$

Question 4. Let $\mathscr{W} = \{p(x) = a_0 + a_2x^2 + a_3x^3 \mid p(-1) = 0\}$ be a vector subspace of P_3.

a. *(4 marks)* Find a basis S for $\mathscr{W}$.

b. *(2 marks)* Determine the dimension of $\mathscr{W}$, Justify.

c. *(2 marks)* Find the coordinate vector of $p(x) = -2 + 2x^2$ relative to the basis S.

a) Let $p(x) = a_0 + a_2x^2 + a_3x^3$ if $p(-1)=0$
$0 = a_0 + a_2 - a_3$
$a_3 = a_0 + a_2$

so $p(x) = a_0 + a_2x^2 + (a_0+a_2)x^3 \in W$

$p(x) = a_0\underbrace{(1+x^3)}_{P_1} + a_2\underbrace{(x^2+x^3)}_{P_2}$ So $W = \text{span}(\{P_1,P_2\})$

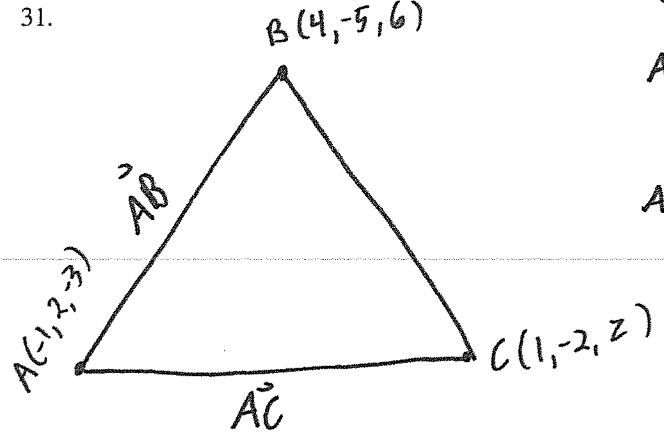
$\{P_1,P_2\}$ are l.i. since they are not a multiple of each other
So $S = \{P_1,P_2\}$ is a basis for W.

b) $\dim(W) = 2$ since the basis has 2 elements.

c) $(-2+2x^2)_S = (c_1,c_2) = (-2,2)$ since

$-2+2x^2 = c_1(1+x^3) + c_2(x^2+x^3)$
$= (-2)(1+x^3) + (2)(x^2+x^3)$

Question 5. (4 marks) Find all values of z such that the triangle with vertices $A(-1,2,-3)$, $B(4,-5,6)$ and $C(1,-2,z)$ has area equal to 31.



$$\vec{AB} = B - A$$
$$= (4,-5,6) - (-1,2,-3) = (5,-7,9)$$
$$\vec{AC} = C - A$$
$$= (1,-2,z) - (-1,2,-3)$$
$$= (2,-4,z+3)$$

$$\vec{AB} \times \vec{AC} = \left(\begin{vmatrix} -7 & 9 \\ -4 & z+3 \end{vmatrix}, - \begin{vmatrix} 5 & 9 \\ 2 & z+3 \end{vmatrix}, \begin{vmatrix} 5 & -7 \\ 2 & -4 \end{vmatrix} \right)$$
$$= \left(-7(z+3)+36, -5(z+3)+18, -20+14 \right)$$
$$= (-7z+15, -5z+3, -6)$$

$$31 = \text{Area}$$
$$31 = \|\vec{AB} \times \vec{AC}\|/2$$
$$31 = \sqrt{(-7z+15)^2 + (-5z+3)^2 + (-6)^2}/2$$
$$62^2 = (49z^2 - 210z + 225) + (25z^2 - 30z + 9) + 36$$
$$0 = 74z^2 + 240z - 3574$$
$$z = \frac{-240 \pm \sqrt{(240)^2 - 4(74)(-3574)}}{2(74)}$$
$$= \frac{-240 \pm \sqrt{1115304}}{148}$$

Question 6. *(3 marks)* Prove: If V be a vectorspace, $\vec{0}$ the zero vector of V, k a scalar then $k\vec{0} = \vec{0}$. **Show EVERY step and justify EVERY step with axiom names when an axiom is used, you may use the fact that $0\vec{u} = \vec{0}$.**

$LHS = k\vec{0}$

$= k(0\vec{0})$ from the fact $0\vec{u} = \vec{0}$ holds for all $\vec{u}$

$= (k0)\vec{0}$ associativity of scalar multiplication

$= 0 \cdot \vec{0}$

$= \vec{0}$ from the fact $0\vec{u} = \vec{0}$

$= RHS$

Question 7. *(4 marks)* Given

$$\vec{u}_1 = (1,2,3), \quad \vec{u}_2 = (3,2,K), \quad \vec{v}_1 = (1,-2,3), \quad \vec{v}_2 = (2,3,t)$$

find all values of t, k such that span($\{\vec{u}_1, \vec{u}_2\}$) $\subseteq$ span($\{\vec{v}_1, \vec{v}_2\}$)

So the span is contained if $u_i \in span(\{\vec{v}_1, \vec{v}_2\})$

$u_1 = c_1 v_1 + c_2 v_2$

$(1,2,3) = c_1(1,-2,3) + c_2(2,3,t)$

$1 = c_1 + 2c_2 \Rightarrow c_1 = 1 - 2c_2$ (1)

$2 = -2c_1 + 3c_2$ (2)

$3 = 3c_1 + c_2 t$ (3)

sub (1) into (2)

$2 = -2(1-2c_2) + 3c_2$

$2 = -2 + 4c_2 + 3c_2$

$4 = 7c_2$

$\frac{4}{7} = c_2$

and sub (1) $c_1 = 1 - \frac{8}{7} = \frac{-1}{7}$

sub into (3)

$3 = 3\left(\frac{-1}{7}\right) + \frac{4}{7}t$

$3 + \frac{3}{7} = \frac{4}{7}t$

$21 + 3 = 4t$

$24 = 4t$

$6 = t$

$u_2 = c_1' v_1 + c_2' v_2$

$(3,2,K) = c_1'(1,-2,3) + c_2'(2,3,t)$

$(3,2,K) = c_1'(1,-2,3) + c_2'(2,3,6)$

$3 = c_1' + 2c_2' \Rightarrow c_1' = 3 - 2c_2'$ (1)

$2 = -2c_1' + 3c_2'$ (2)

$K = 3c_1' + 6c_2'$ (3)

sub (1) into (2) $2 = -2(3-2c_2') + 3c_2'$

$2 = -6 + 7c_2'$

$\frac{8}{7} = c_2'$ sub into (1)

$c_1' = 3 - 2\left(\frac{8}{7}\right) = \frac{5}{7}$

sub into (3) $K = 3\left(\frac{5}{7}\right) + 6\left(\frac{8}{7}\right) = 9$

$\therefore t = 6$ and $K = 9$

Bonus Question. *(5 marks)* In a vector space, the span of any subset is a subspace. In addition, the span of a subset of a vector space is the smallest subspace containing all vectors of the subset.

Theorem 2.3. *In a vector space, the span of any subset is a subspace. In addition, the span of a subset of a vector space is the smallest subspace containing all vectors of the subset.*

Lets show its a subspace by applying the subspace test.

closure under addition: let $\left.\begin{array}{l}\vec{v} = c_1\vec{s}_1 + c_2\vec{s}_2 + \dots + c_n\vec{s}_n, \\ \vec{u} = k_1\vec{s}_1 + k_2\vec{s}_2 + \dots + k_n\vec{s}_n\end{array}\right\} \in \text{span}(S)$

then $\vec{v} + \vec{u} = c_1\vec{s}_1 + c_2\vec{s}_2 + \dots + c_n\vec{s}_n + k_1\vec{s}_1 + k_2\vec{s}_2 + \dots + k_n\vec{s}_n$

$= (c_1+k_1)\vec{s}_1 + (c_2+k_2)\vec{s}_2 + \dots + (c_n+k_n)\vec{s}_n \in \text{span}(S)$ since it is a linear combination of $\{\vec{s}_1, \dots, s_n\}$.

closure under scalar mult.: let $\vec{v} = c_1\vec{s}_1 + c_2\vec{s}_2 + \dots + c_n\vec{s}_n \in \text{span}(S)$ and $r \in \mathbb{R}$

then $r\vec{v} = r(c_1\vec{s}_1 + c_2\vec{s}_2 + \dots + c_n\vec{s}_n) = rc_1\vec{s}_1 + rc_2\vec{s}_2 + \dots + rc_n\vec{s}_n \in \text{span}(S)$ since it is a linear combination of $\{\vec{s}_1, \dots, \vec{s}_n\}$

$\therefore$ span(S) is a subspace

Suppose $\exists\, W \subset \text{span}(S)$ s.t. $\forall s_i \in S$, $s_i \in W$ and $\vec{v} \in W$ but $v \notin \text{span}(S)$

Since $W \subset \text{span}(S)$ then $\vec{v} = c_1\vec{s}_1 + \dots + c_n\vec{s}_n$ $\lightning$

Since span(S) is the set of all linear combination of S.

$\therefore$ Span(S) is the smallest subspace.