Dawson College: **Linear Algebra**: 201-NYC-05-S06: Winter 2016

Name: Y. Lamontagne

Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1.

a. *(8 marks)* Solve for all a, b, c, d such that

$$\begin{bmatrix} 3a+3b+7c-3d & 2a+3b+3c+d \\ 4a+17c-2d & 9a+6b+27c-4d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

b. *(1 mark)* Find two particular solution $(a,\ b,\ c,\ d)$ that satisfy the above matrix equation.

c. *(1 mark)* Find a solution to the above matrix equation where $c = 1$.

$$\begin{aligned} 3a+3b+7c-3d &= 1 \\ 2a+3b+3c+d &= 2 \\ 4a+17c-2d &= 3 \\ 9a+6b+27c-4d &= 6 \end{aligned}$$

$$\begin{bmatrix} 3 & 3 & 7 & -3 & 1 \\ 2 & 3 & 3 & 1 & 2 \\ 4 & 0 & 17 & -2 & 3 \\ 9 & 6 & 27 & -4 & 6 \end{bmatrix}$$

$$\sim \begin{matrix} \\ 3R_2 \to R_2 \\ 3R_3 \to R_3 \\ \\ \end{matrix} \begin{bmatrix} 3 & 3 & 7 & -3 & 1 \\ 6 & 9 & 9 & 3 & 6 \\ 12 & 0 & 51 & -6 & 9 \\ 9 & 6 & 27 & -4 & 6 \end{bmatrix}$$

$$\sim \begin{matrix} \\ -2R_1+R_2 \to R_2 \\ -4R_1+R_3 \to R_3 \\ -3R_1+R_4 \to R_4 \end{matrix} \begin{bmatrix} 3 & 3 & 7 & -3 & 1 \\ 0 & 3 & -5 & 9 & 4 \\ 0 & -12 & 23 & 6 & 5 \\ 0 & -3 & 6 & 5 & 3 \end{bmatrix}$$

$$\sim \begin{matrix} \\ \\ 4R_2+R_3 \to R_3 \\ R_2+R_4 \to R_4 \end{matrix} \begin{bmatrix} 3 & 3 & 7 & -3 & 1 \\ 0 & 3 & -5 & 9 & 4 \\ 0 & 0 & 3 & 42 & 21 \\ 0 & 0 & 1 & 14 & 7 \end{bmatrix}$$

$$\sim \begin{matrix} \\ \\ \frac{1}{3}R_3 \to R_3 \\ -\frac{1}{3}R_3+R_4 \to R_4 \end{matrix} \begin{bmatrix} 3 & 3 & 7 & -3 & 1 \\ 0 & 3 & -5 & 9 & 4 \\ 0 & 0 & 1 & 14 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} -7R_3+R_1 \to R_1 \\ 5R_3+R_2 \to R_2 \end{matrix} \begin{bmatrix} 3 & 3 & 0 & -101 & -48 \\ 0 & 3 & 0 & 79 & 39 \\ 0 & 0 & 1 & 14 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim -R_2+R_1 \to R_1 \begin{bmatrix} 3 & 0 & 0 & -180 & -87 \\ 0 & 3 & 0 & 79 & 39 \\ 0 & 0 & 1 & 14 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} \frac{1}{3}R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & -60 & -29 \\ 0 & 1 & 0 & 79/3 & 13 \\ 0 & 0 & 1 & 14 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $d = t$, $t \in \mathbb{R}$

$\therefore (a,b,c,d) = \left(-29+60t,\ 13-\frac{79}{3}t,\ 7-14t,\ t\right)$

b) $t=0$ $(-29, 13, 7, 0)$

$t=3$ $(151, -66, -35, 3)$

c) $c=1 \Rightarrow 7-14t=1$

$6 = 14t$

$\frac{3}{7} = t$

$\left(-29+60\left(\frac{3}{7}\right),\ 13-\frac{79}{3}\cdot\frac{3}{7},\ 1,\ \frac{3}{7}\right)$

$= \left(-\frac{23}{7},\ \frac{12}{7},\ 1,\ \frac{3}{7}\right)$

Question 2. Consider the matrices:

$$A=[a_{ij}]_{3\times 3},\quad B=[b_{ij}]_{2\times 3},\quad C=[c_{ij}]_{3\times 2},\quad D=[d_{ij}]_{2\times 2},\quad E=[e_{ij}]_{3\times 6},$$

where $a_{ij}=i-j$, $b_{ij}=(-1)^i 2+(-1)^j 3$, $c_{ij}=i+j$, $d_{ij}=(ij)^2$, $e_{ij}=i+j$. Evaluate the following if possible, justify.

a. *(2 marks)* EA

$E_{3\times 6} A_{3\times 3}$ not defined since # rows of A ≠ # columns of E.

b. *(2 marks)* AB^T $= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -5 & 1 & -5 \\ -1 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -5 & -1 \\ 1 & 5 \\ -5 & -1 \end{bmatrix}$

$= \begin{bmatrix} 9 & -3 \\ 0 & 0 \\ -9 & 3 \end{bmatrix}$

c. *(2 marks)* C^TAB^T $= \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}^T \begin{bmatrix} 9 & -3 \\ 0 & 0 \\ -9 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ 0 & 0 \\ -9 & 3 \end{bmatrix} = \begin{bmatrix} -18 & 6 \\ -18 & 6 \end{bmatrix}$

d. *(2 marks)* $\text{trace}(D^2)$

$D^2 = \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}^2 = \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix} = \begin{bmatrix} 17 & 68 \\ 68 & 272 \end{bmatrix}$

$\text{trace}(D^2) = 17 + 272 = 289$

e. *(2 marks)* $(3BC-2D)^T$ $= \left(3 \begin{bmatrix} -5 & 1 & -5 \\ -1 & 5 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} - 2 \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}\right)^T$

$= \left(3 \begin{bmatrix} -27 & -36 \\ 9 & 12 \end{bmatrix} - 2 \begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}\right)^T = \begin{bmatrix} -83 & -116 \\ 19 & 4 \end{bmatrix}^T = \begin{bmatrix} -83 & 19 \\ -116 & 4 \end{bmatrix}$

Question 3. Consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

a. *(5 marks)* Determine A^{-1}.

b. *(2 marks)* Determine $((2A)^T)^{-1}$, if possible.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{-R_2+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\underset{-R_3 \to R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\underset{\substack{-3R_3+R_1 \to R_1 \\ -2R_3+R_2 \to R_2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & -3 & 3 \\ 0 & 2 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\underset{\substack{-R_2+R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1/2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

b) $$((2A)^T)^{-1} = (2A^T)^{-1} = \frac{1}{2}(A^T)^{-1}$$

$$= \frac{1}{2}(A^{-1})^T$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ 0 & -1/2 & 1 \\ 0 & 1 & -1 \end{bmatrix}^T = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ -2 & -1/2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ -1 & -1/4 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

Question 4. *(2 marks)* Find a formula for trace(A^{-1}) where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if A is invertible

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{bmatrix}$$

$$\text{trace}(A^{-1}) = \frac{a+d}{(ad-bc)}$$

Question 5. *(3 marks)* Prove: If E_i are elementary matrices and $E_n \cdots E_2E_1A$ is invertible then A is invertible.

If $E_n \cdots E_2E_1A$ is invertible then by TFAE it is expressible as a product of elementary matrices i.e.

$$E_n \cdots E_2E_1A = F_1F_2\cdots F_K \quad \text{where } F_i \text{ are elem. matrices}$$

E_i are elem. matrices. So it follows that

$$(E_n \cdots E_2E_1)^{-1} E_n \cdots E_2E_1A = (E_n \cdots E_2E_1)^{-1} F_1F_2 \cdots F_K$$

$$IA = E_1^{-1}E_2^{-1} \cdots E_n^{-1} F_1F_2 \cdots F_K$$

$$A = E_1^{-1}E_2^{-1} \cdots E_n^{-1} F_1F_2 \cdots F_K$$

Since A is expressible as a product of elem. matrices then A is invertible by TFAE.

Question 6. *(3 marks)* Given a matrix A which satisfy $A^3+3A^2+A+I=0$ find the inverse of A in terms of A and the identity.

It follows that

$$-A^3 - 3A^2 - A = I$$

and

$$A(-A^2 - 3A - I) = I$$

$$(-A^2 - 3A - I)A = I$$

$\therefore$ by definition $A^{-1} = -A^2 - 3A - I$.

Question 7. *(5 marks)* Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$(X^T E_1 E_2 E_3)^T = A$$

solve for X, if possible.

$$E_1^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^T = E_2 \quad E_3^T = E_3$$

$$E_3^T E_2^T E_1^T (X^T)^T = A$$

$$(E_3^T E_2^T E_1^T)^{-1}(E_3^T E_2^T E_1^T) X = (E_3^T E_2^T E_1^T)^{-1} A$$

$$X = (E_1^T)^{-1}(E_2^T)^{-1}(E_3^T)^{-1} A$$

$$X = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

$-2R_3 + R_1 \to R_1$ $\quad -\frac{1}{3}R_2 \to R_2$ $\quad R_1 \leftrightarrow R_3$

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2/3 & -2/3 & -2/3 & -2/3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Bonus Question. *(5 marks)*
Given

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

where the entries of the matrix are elements of $\mathbb{Z}_3$. Operations on the elements of $\mathbb{Z}_3$ can be defined by the following Cayley tables:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Express A as a product of elementary matrices, if possible.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{matrix} -R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & 2 \end{bmatrix} \sim -2R_2 + R_3 \to R_3 \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \sim -2R_3 + R_2 \to R_2 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim 2R_2 \to R_2 \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim -2R_2 + R_1 \to R_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_6 E_5 E_4 E_3 E_2 E_1 A = I$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$