**Dawson College**: **Linear Algebra**: 201-NYC-05-S06: Winter 2016

**Name:** Y. Lamontagne

# Test 2

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

**Question 1.** Given

$$A = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & -4 & 5 & 6 \\ -7 & 0 & 0 & 0 \\ 8 & 10 & 11 & 0 \end{bmatrix}.$$

a. *(5 marks)* Evaluate $\det(A)$.

b. *(3 marks)* If $B$ is a $4 \times 4$ matrix and $\det(2BA + B\operatorname{adj}(A)A^2) = 0$ then $B$ singular.

a) $\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} + a_{34}C_{34}$

$= -7(-1)^{3+1}\begin{vmatrix} 0 & 2 & -3 \\ -4 & 5 & 6 \\ 10 & 11 & 0 \end{vmatrix} + 0C_{32} + 0C_{33} + 0C_{34}$

$= -7\left[a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}\right]$

$= -7\left[0C_{11} + 2(-1)^{1+2}\begin{vmatrix} -4 & 6 \\ 10 & 0 \end{vmatrix} + (-3)(-1)^{1+3}\begin{vmatrix} -4 & 5 \\ 10 & 11 \end{vmatrix}\right]$

$= -7\left[-2[-60] - 3[-94]\right] = -2814$

b) $0 = \det(2BA + B\operatorname{adj}(A)A^2)$ $\qquad A^{-1} = \frac{1}{\det A}\operatorname{adj} A$

$0 = \det(B(2A + \operatorname{adj}(A)A^2))$

$0 = \det B \det(2A + \operatorname{adj}(A)A^2)$ $\qquad \det A\, A^{-1} = \operatorname{adj} A$

$0 = \det B \det(2A + \det A\, \underbrace{A^{-1}A}_{I} \cdot A)$

$0 = \det B \det(2A + \det A\, A)$

$0 = \det B \det((2 + \det A)A)$

$0 = \det B\, \underbrace{(2 + \det A)^4}_{\neq 0} \underbrace{\det A}_{\neq 0}$

$\therefore \det B = 0 \quad \therefore B$ is singular.

**Question 2.** (2 marks) Prove or disprove: If $\det(A^2) = \det(A)$ then $A^2 = A$.

disprove, Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\det A = 0 = \det A^2$ but $A^2 \neq A$

**Question 3.** *(3 marks)* Prove: If $A$ is an $n \times n$ skew-symmetric matrix where $n$ is odd then $\det(A) = 0$.

If $A$ is skew symmetric then $A^T = -A$

$$\det(A^T) = \det(-A)$$
$$\det A = (-1)^n \det(A) \quad \text{since } n \text{ is odd}$$
$$\det A = -\det A$$
$$\therefore \det A = 0$$

**Question 4.** *(2 marks)* Show that there exist an $n \times n$ symmetric elementary matrix $E$ such that $E^2 = I$.

$$I_n \sim R_1 \leftrightarrow R_n \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} = E$$

$E^T = E$ and $E^2 = EE = I$

interchanges $R_1$ and $R_n$ of $E$.

**Question 5.** *(3 marks)* Prove or disprove: If a system of linear equations $Ax = b$ has $n$ equations and $n$ variables has infinitely many solutions then $Ax = c$ has infinitely many solutions for all $n \times 1$ matrix $c$.

disprove,

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

then $Ax = b$ has augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ which has infinitely many solutions

And $Ax = c$ has augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right]$ which has no solutions.

**Question 6.** *(5 marks)* Given

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & \frac{3}{a} & \frac{3}{a} \\ 0 & 0 & 0 & \frac{2}{b} \\ 0 & \frac{1}{c} & 0 & \frac{1}{c} \\ \frac{4}{d} & 0 & 0 & \frac{4}{d} \end{bmatrix}$$

If $\det(B) = 5$ then determine $\det(A)$.

$$B \overset{-C_3+C_4 \to C_4}{\sim} \begin{bmatrix} 0 & 0 & 3/a & 0 \\ 0 & 0 & 0 & 2/b \\ 0 & 1/c & 0 & 1/c \\ 4/d & 0 & 0 & 4/d \end{bmatrix} \overset{-C_2+C_4 \to C_4}{\sim} \begin{bmatrix} 0 & 0 & 3/a & 0 \\ 0 & 0 & 0 & 2/b \\ 0 & 1/c & 0 & 0 \\ 4/d & 0 & 0 & 4/d \end{bmatrix} \overset{-C_1+C_4 \to C_4}{\sim} \begin{bmatrix} 0 & 0 & 3/a & 0 \\ 0 & 0 & 0 & 2/b \\ 0 & 1/c & 0 & 0 \\ 4/d & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{C_1 \leftrightarrow C_4,\ C_2 \leftrightarrow C_3}{\sim} \begin{bmatrix} 0 & 3/a & 0 & 0 \\ 3/b & 0 & 0 & 0 \\ 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 4/d \end{bmatrix} \overset{C_1 \leftrightarrow C_2}{\sim} \begin{bmatrix} 3/a & 0 & 0 & 0 \\ 0 & 2/b & 0 & 0 \\ 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 4/d \end{bmatrix}$$

$$\overset{\frac{1}{3}R_1 \to R_1,\ \frac{1}{2}R_2 \to R_2,\ \frac{1}{4}R_4 \to R_4}{\sim} \begin{bmatrix} 1/a & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1/d \end{bmatrix} = C$$

$$(-1)(-1)(-1)(\tfrac{1}{3})(\tfrac{1}{2})(\tfrac{1}{4}) \det B = \det C$$

$$\frac{-5}{4!} = \det C$$

Notice $C^{-1} = A$

$$\det C^{-1} = \det A$$

$$\det A = \frac{1}{\det C}$$

$$\det A = \frac{1}{\det C} = \frac{-4!}{5}$$

**Question 7.** Given $\vec{u}, \vec{v} \in \mathbb{R}^2$ where $\vec{u} = (1, 1)$, $||\vec{v}|| = 1$ and the angle between $\vec{u}$ and $\vec{v}$ is $\pi/4$

a. *(2 marks)* Determine $\vec{u} \cdot \vec{v}$.

b. *(3 marks)* Determine $\vec{v}$.

a) $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta = \|(1,1)\| \, 1 \, \cos \pi/4 = \sqrt{1^2+1^2} \, 1 \, \frac{1}{\sqrt{2}} = 1$

b) Let $\vec{v} = (a, b)$, $\vec{u} \cdot \vec{v} = 1$

$$1 = (1,1) \cdot (a,b)$$
$$1 = a + b$$
$$b = 1 - a$$

$$\|(a,b)\| = 1$$
$$\sqrt{a^2+b^2} = 1$$
$$a^2+b^2 = 1$$
$$1 = a^2 + (1-a)^2$$
$$1 = a^2 + 1 - 2a + a^2$$
$$0 = 2a^2 - 2a$$
$$0 = a(a-1)$$
$$a = 0 \quad a = 1$$

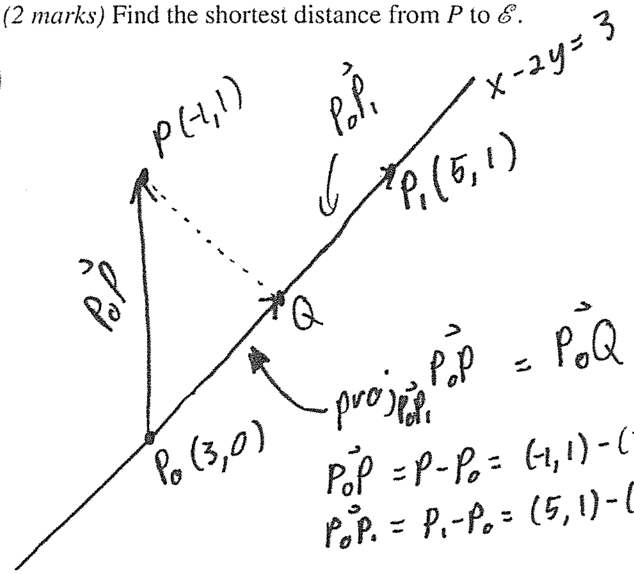
$\therefore \vec{v} = (0, 1)$ and $\vec{v} = (1, 0)$

**Question 8.** Given $\mathscr{E} : x - 2y = 3$ and $P = (-1, 1)$.

a. *(5 marks)* Find the closest point on $\mathscr{E}$ from $P$.

b. *(2 marks)* Find the shortest distance from $P$ to $\mathscr{E}$.

a)




$\vec{P_0P} = P - P_0 = (-1,1) - (3,0) = (-4,1)$

$\vec{P_0P_1} = P_1 - P_0 = (5,1) - (3,0) = (2,1)$

Let $y = 0$: $x - 2(0) = 3$, $x = 3$ $\therefore P_0(3,0)$

Let $y = 1$: $x - 2(1) = 3$, $x = 5$ $\therefore P_1(5,1)$

So $\vec{P_0Q} = \text{proj}_{\vec{P_0P_1}} \vec{P_0P}$

$$Q - P_0 = \frac{\vec{P_0P} \cdot \vec{P_0P_1}}{\vec{P_0P_1} \cdot \vec{P_0P_1}} \vec{P_0P_1}$$

$$Q = P_0 + \frac{(-4,1)\cdot(2,1)}{(2,1)\cdot(2,1)}(2,1)$$

$$= (3,0) + \frac{-7}{5}(2,1) = \left(\frac{1}{5}, \frac{-7}{5}\right)$$

closest point

b) $\|\vec{PQ}\|$ = distance

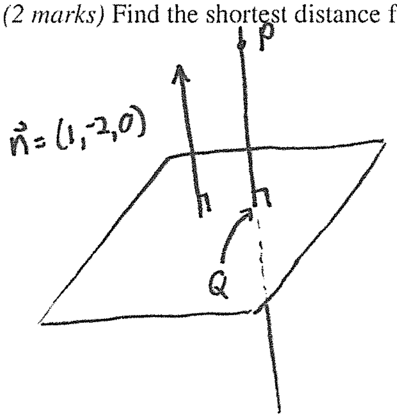
distance $= \|Q - P\|$
$$= \left\| \left(\frac{1}{5}, \frac{-7}{5}\right) - (-1,1) \right\|$$
$$= \left\| \left(\frac{6}{5}, \frac{-12}{5}\right) \right\|$$
$$= \frac{6}{5} \|(1,-2)\|$$
$$= \frac{6}{5}\sqrt{1^2+(-2)^2}$$
$$= \frac{6\sqrt{5}}{5}$$

**Question 9.** Given $\mathscr{E} : x - 2y = 3$ and $P = (-1, 1, -1)$.

a. *(2 marks)* Find the equation of a line orthogonal to $\mathscr{E}$ that passes through $P$.

b. *(2 marks)* Find the closest point on $\mathscr{E}$ from $P$.

c. *(2 marks)* Find the shortest distance from $P$ to $\mathscr{E}$.




a) $\vec{x} = (-1,1,-1) + t\,(1,-2,0) \quad t \in \mathbb{R}$

$$\mathscr{L} : \begin{cases} x = -1 + t \\ y = 1 - 2t \\ z = -1 \end{cases}$$

b) Intersection of $\mathscr{L}$ and $\mathcal{E}$ to find the closest point

$$(-1+t) - 2(1-2t) = 3$$
$$-1 + t - 2 + 4t = 3$$
$$5t = 6$$
$$t = 6/5$$

$$Q = (-1,1,-1) + \tfrac{6}{5}(1,-2,0) = \left(\tfrac{1}{5}, \tfrac{-7}{5}, -1\right)$$

c) distance $= \|\vec{PQ}\|$

$= \|Q - P\|$

$= \left\|\left(\tfrac{1}{5}, \tfrac{-7}{5}, -1\right) - (-1,1,-1)\right\|$

$= \left\|\left(\tfrac{6}{5}, -\tfrac{12}{5}, 0\right)\right\| = \dfrac{6\sqrt{5}}{5}$

**Bonus Question.** *(5 marks)*

$$\operatorname{adj}(\operatorname{adj}(A)) = (\det(A))^{n-2}A$$

$\square^{-1} = \dfrac{1}{\det \square}\operatorname{adj}\square$

$\operatorname{adj}\square = (\det\square)\square^{-1}$

so $\operatorname{adj} A = (\det A)A^{-1}$

and $\operatorname{adj}(\det A\, A^{-1}) = (\det((\det A)A^{-1}))\cdot(\det A^{-1})^{-1}$

LHS $= \operatorname{adj}(\operatorname{adj}(A))$

$= \operatorname{adj}((\det A)A^{-1})$

$= (\det((\det A)A^{-1}))\cdot((\det A)A^{-1})^{-1}$

$= (\det A)^n \det A^{-1} \dfrac{1}{\det A}(A^{-1})^{-1}$

$= (\det A)^n \dfrac{1}{\det A}\dfrac{1}{\det A} A = (\det A)^{n-2} A$

$\operatorname{adj}(\operatorname{adj} A) = (10!)^8 A$