Dawson College: **Linear Algebra**: 201-NYC-05-S06: Winter 2016

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Test 3

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given the vectors $\vec{u} = (\lambda + 1, 1, \lambda)$, $\vec{v} = (\lambda, 2, 2)$ and $\vec{w} = (1, 1, \lambda\quad)$.

a. *(3 marks)* For which value(s) of λ if any, are the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ linearly independent.

b. *(1 mark)* For which value(s) of λ if any, $\text{span}(\{\vec{u}, \vec{v}, \vec{w}\}) = \mathbb{R}^3$.

c. *(2 marks)* For which value(s) of λ if any, $\text{span}(\{\vec{u}, \vec{v}, \vec{w}\})$ is a plane through the origin.

d. *(1 mark)* For which value(s) of λ if any, $\text{span}(\{\vec{u}, \vec{v}, \vec{w}\})$ is a line through the origin.

e. *(3 marks)* For which value(s) of λ if any, the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ generate a parallelpiped of volume 2016.

a) If $\vec{u}\cdot(\vec{v}\times\vec{w}) \neq 0$ then the vectors are l.i.

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} \lambda+1 & 1 & \lambda \\ \lambda & 2 & 2 \\ 1 & 1 & \lambda \end{vmatrix}$$

$$= (\lambda+1)(-1)^{1+1}\begin{vmatrix} 2 & 2 \\ 1 & \lambda \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} \lambda & 2 \\ 1 & \lambda \end{vmatrix} + \lambda(-1)^{1+3}\begin{vmatrix} \lambda & 2 \\ 1 & 1 \end{vmatrix}$$

$$= (\lambda+1)[2(\lambda\ ) - (\ 2)] - \lambda(\lambda\ ) + (2) + \lambda(\lambda-2)$$

$$= 2(\lambda+1)(\lambda-1) - \lambda^2 + 2 + \lambda^2 - 2\lambda$$

$$= 2\lambda^2 - 2 - \lambda^2 + 2 + \lambda^2 - 2\lambda$$

$$= 2\lambda^2 - 2\lambda$$

So $\vec{u}\cdot(\vec{v}\times\vec{w}) \neq 0$

$2\lambda^2 - 2\lambda \neq 0$

$2\lambda(\lambda - 1) \neq 0$

$\lambda \neq 0$, $\lambda \neq 1$

b) For $\lambda \neq 0, 1$ $\{\vec{u}, \vec{v}, \vec{w}\}$ is l.i. and $\dim(\mathbb{R}^3) = 3$ it follows by a theorem seen in class that $\{\vec{u}, \vec{v}, \vec{w}\}$ spans $\mathbb{R}^3$. ∴ a basis of $\mathbb{R}^3$

c) if $\lambda = 0$ then $\vec{u} = (1,1,0)$, $\vec{v} = (0,2,2)$, $\vec{w} = (1,1,0)$ and $\vec{u} \neq k\vec{v}$ for any k and $\vec{u} = \vec{w}$

Hence $\dim(\text{span}(\{\vec{u}, \vec{v}, \vec{w}\})) = 2$. So it spans a plane.

if $\lambda = 1$ then $\vec{u} = (2,1,1)$, $\vec{v} = (1,2,2)$, $\vec{w} = (1,1,1)$ and $\vec{u} \neq k\vec{v}$ for any k and $\frac{1}{3}\vec{u} + \frac{1}{3}\vec{v} = \vec{w}$

Hence $\dim(\text{span}(\{\vec{u}, \vec{v}, \vec{w}\})) = 2$. So it spans a plane

d) by c, d it follows that there are no values of λ such that the span of $\vec{u}, \vec{v}, \vec{w}$ span a line through the origin

e) Volume of parallelpiped $= |\vec{u}\cdot(\vec{v}\times\vec{w})|$

$2016 = |2\lambda^2 - 2\lambda|$

$1008 = |\lambda^2 - \lambda|$

$-1008 = \lambda^2 - \lambda$

$0 = \lambda^2 - \lambda + 1008$

$\lambda = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(1008)}}{2(1)}$ = no sol.

$1008 = \lambda^2 - \lambda$

$0 = \lambda^2 - \lambda - 1008$

$\lambda = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1008)}}{2(1)} = \dfrac{1 \pm \sqrt{4033}}{2}$

Question 2. (2 marks) Determine whether the statement is true or false, and justify your answer.
The general solution of the nonhomogeneous linear system $Ax = b$ can be obtained by adding b to the general solution of the homogeneous linear system $Ax = 0$.

False,

Given $\underset{A}{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}} \underset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

The solution set of $Ax = b$ $\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

is $(x, y) = (0, t)$ $t \in \mathbb{R}$

and $(x, y) = b + (0, t)$
$= (1, 0) + (0, t)$

is not solution set of $Ax = b$.

Question 3. *(6 marks)* Given the lines: $\mathscr{L}_1 : \vec{x} = \underbrace{(1,2,0)}_{P_1} + t\underbrace{(-1,0,1)}_{\vec{d}_1}$, $t \in \mathbb{R}$ and $\mathscr{L}_2 : \vec{x} = \underbrace{(1,3,-1)}_{P_2} + s\underbrace{(2,1,0)}_{\vec{d}_2}$, $s \in \mathbb{R}$. Find the shortest distance between $\mathscr{L}_1$ and $\mathscr{L}_2$.

$\mathscr{L}_1$ and $\mathscr{L}_2$ are not parallel since $\vec{d}_1 \neq k\vec{d}_2$ for any k.

$\mathscr{L}_1 : \begin{array}{l} x = 1 - t \\ y = 2 \\ z = t \end{array}$ $\quad \mathscr{L}_2 : \begin{array}{l} x = 1 + 2s \\ y = 3 + s \\ z = -1 \end{array}$

Lets determine whether $\mathscr{L}_1$ and $\mathscr{L}_2$ intersect

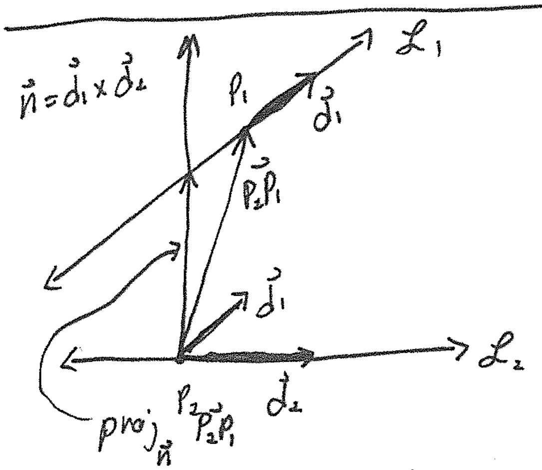
① $1 - t = 1 + 2s$
② $2 = 3 + s \Rightarrow s = -1$
③ $t = -1$

sub ② and ③ into ① to check for consistency

$1 - (-1) \overset{?}{=} 1 + 2(-1)$
$2 \neq -1$

$\therefore \mathscr{L}_1$ and $\mathscr{L}_2$ do not intersect.

$\therefore \mathscr{L}_1$ and $\mathscr{L}_2$ are skew lines.



$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \left(\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} \right)$

$\begin{array}{cc} -1 & 2 \\ 0 & 1 \\ 1 & 0 \end{array} \quad = (-1, 2, -1)$

$\vec{P_2P_1} = P_1 - P_2$
$= (1, 2, 0) - (1, 3, -1)$
$= (0, -1, 1)$

distance $= \| \text{proj}_{\vec{n}} \vec{P_2P_1} \|$

$= \left\| \frac{\vec{n} \cdot \vec{P_2P_1}}{\vec{n} \cdot \vec{n}} \vec{n} \right\| = \left\| \frac{(-1,2,-1) \cdot (0,-1,1)}{(-1,2,-1) \cdot (-1,2,-1)} (-1,2,-1) \right\| = \left\| \frac{-2-1}{1+4+1} (-1,2,-1) \right\|$

$= \left\| \frac{-3}{6} (-1, 2, -1) \right\|$

$= \frac{1}{2} \sqrt{(-1)^2 + 2^2 \; (-1)^2}$

$= \frac{\sqrt{6}}{2}$

Question 4. *(4 marks)* Determine whether

$$W = \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x)\}$$

is a subspace of the vector space of real functions.

Let $f, g \in W$, it follows that $f(-x) = -f(x)$ and $g(-x) = -g(x)$

$f+g \in W$ since $(f+g)(-x) = f(-x) + g(-x)$
$= -f(x) - g(x)$
$= -(f(x)+g(x)) = -(f+g)(x)$

$\therefore$ closed under vector addition

Let $f \in W$, it follows that $f(-x) = -f(x)$, $r \in \mathbb{R}$

$rf \in W$ since $(rf)(-x) = r f(-x) = r(-f(x)) = -rf(x)$

$\therefore$ closed under scalar multiplication.

$\therefore$ a subspace by the subspace test.

Question 5. *(2 marks)* Determine whether the following is a vector space $V = \{A \mid A \in \mathcal{M}_{2\times 2} \text{ and } A^T = A\}$ with the following operations:

$A + B = AB$ and $kA = kA$

That is, vector addition is matrix multiplication and scalar multiplication is the regular scalar multiplication.. Justify.

The above is not a vector space since it is not closed under addition.

Let $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \in V$

the $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \notin V.$

Question 6. Given the set

$$S = \{1+x+x^2, 1+x^3\}$$

a. *(2 marks)* Determine whether S is linearly independent.

b. *(4 marks)* Find a new set S' that contains the elements of S and is a basis for $\mathscr{P}_3$. Show that S' is a basis.

c. *(2 marks)* Express $p(x) = x + x^2 - x^3$ relative to the basis S'.

a) Let $p_1(x) = 1 + x + x^2$ and $p_2(x) = 1 + x^3$, since $p_1(x)$ is not a multiple of $p_2(x)$ the set is linearly independent.

b) $\dim(\mathscr{P}_3) = 4$, for S' to be a basis of $\mathscr{P}_3$ it needs to contain 4 linearly independent vectors.

Let $S' = \{1, x^2, 1+x+x^2, 1+x^3\}$ and lets show that S' is linearly independent.

$0 + 0x + 0x^2 + 0x^3 = c_1(1) + c_2(x^2) + c_3(1+x+x^2) + c_4(1+x^3)$

$0 + 0x + 0x^2 + 0x^3 = (c_1 + c_2 + c_3 + c_4) + c_3(x) + (c_2 + c_3)x^2 + c_4 x^3$

So ① $0 = c_1 + c_2 + c_3 + c_4$

② $0 = c_3$

③ $0 = c_2 + c_3$ by ② and ③ $c_2 = 0$ ⑤

④ $0 = c_4$

Sub ②, ④ and ⑤ into ① $c_1 = 0$

$\therefore$ S' is linearly independent

$\therefore$ S' is a basis of $\mathscr{P}_3$

c) $p(x) = x + x^2 - x^3 = c_1(1) + c_2(x^2) + c_3(1+x+x^2) + c_4(1+x^3)$

by inspection $c_4 = -1$, $c_3 = 1$, $c_2 = 0$, $c_1 = 0$

$(p(x))_{S'} = (0, 0, 1, -1)$

Question 7. Given the set

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

a. *(4 marks)* Determine whether S is a basis for the vector space of 2×2 upper triangular matrices with the usual matrix addition and scalar multiplication.

b. *(1 marks)* Determine the dimension of the vectors space of 2×2 upper triangular matrices with the usual matrix addition and scalar multiplication.

$W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$, Let $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in W$

Is $M \in \text{span}(S)$? $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$a = c_1 + c_2 - c_3$
$b = c_1 + 2c_2$
$c = c_1 + c_2 + c_3$

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

If A is invertible then any M is expressible as a linear combination of the matrices of S. $|A| = (-1)\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 0(-1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1(-1)^{1+1}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$

$= 1 + 1 = 2 \neq 0$

$\therefore$ A is invertible

$\therefore$ $M \in \text{span}(S)$ for any a, b, c by TFAE

Lets determine if S is linearly independent.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$0 = c_1 + c_2 - c_3$
$0 = c_1 + 2c_2$
$0 = c_1 + c_2 + c_3$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If A is invertible then the above only has the trivial solution. $\therefore$ S is linearly independent $\therefore$ S is a basis of W

b) $\dim(W) = 3$

Bonus Question. *(5 marks)*

Prove: A subspace of a finite-dimensional vector space is finite-dimensinal.

Prove by contradiction.