Name: ____________________

Test 1

This test is graded out of 45 marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531XG or Sharp EL-531X calculator. Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

Question 1. Given

$$A = \begin{bmatrix} 0 & 2 & -5 & -1 \\ 2 & -6 & 0 & 3 \\ 3 & 0 & 0 & -2 \\ 5 & -4 & -5 & 0 \end{bmatrix}$$

a. *(5 marks)* Find the reduced row echelon form of the matrix A.

b. *(1 mark)* Suppose that A is the augmented matrix of a linear system. Write the corresponding system of linear equations. Use part a. to find the solution of the system.

c. *(2 marks)* Suppose that A is the coefficient matrix of a homogeneous linear system. Write the corresponding system of linear equations. Use part a. to find the solution of the system.

d. *(1 mark)* Suppose that A is the coefficient matrix of a homogeneous linear system. Use part c. to find a solution to the system when $x_1 = -1$.

e. *(2 marks)* Express A as a product of elementary matrices, if possible. Justify.

f. *(1 mark)* **True** or **False**: The reduced row echelon form of any matrix is unique. *(do not justify!)*

Question 2. Consider the matrices:

$$A = \left[a_{ij}\right]_{3\times3}, \quad B = \left[b_{ij}\right]_{2\times3}, \quad C = \left[c_{ij}\right]_{3\times2}, \quad D = \left[d_{ij}\right]_{2\times2}, \quad E = \left[e_{ij}\right]_{3\times6}$$

where $a_{ij} = i - j, \quad b_{ij} = (-1)^i 2 + (-1)^j 3, \quad c_{ij} = i + j, \quad d_{ij} = (ij)^2, \quad e_{ij} = i + j$. Evaluate the following if possible, justify.

a. *(2 marks)* AB

b. *(2 marks)* $B^T E$

Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}, D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Evaluate the following if possible, justify.

a. *(2 marks)* $\operatorname{trace}(C)AB - \operatorname{trace}(D)B$

b. *(2 marks)* $(AB - 2I)A$

Question 3. Given A and B, $n \times n$ matrices.

a. *(2 marks)* Prove or disprove: If $AB = 0$ then $A = 0$ or $B = 0$.

b. *(2 marks)* Prove or disprove: If A is invertible and $AB = 0$ then $B = 0$.

Question 4. Let A and B be $n \times n$ matrices.

a. *(2 marks)* Verify that $A(I+BA) = (I+AB)A$ and that $(I+BA)B = B(I+AB)$.

b. *(3 marks)* Prove: If $I+AB$ is invertible, verify that $I+BA$ is also invertible and that $(I+BA)^{-1} = I - B(I+AB)^{-1}A$. *(Hint: use part a.)*

Question 5.[1] *(5 marks)* The number of leading 1's in a row echelon form of A is called the *rank* of A. Given the following matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & b \\ 0 & 3 & a & b \end{bmatrix}$$

a. Under what conditions on a and b is the rank of A equal to 4.

b. Under what conditions on a and b is the rank of A equal to 3.

c. Under what conditions on a and b is the rank of A equal to 2.

Question 6. Given a 5×5 *Magic Square (matrix)*

$$M = \begin{bmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{bmatrix} \text{ and } A = \begin{bmatrix} 4 & 12 & 25 & 8 & 16 \\ 11 & 24 & 7 & 20 & 3 \\ 34 & 10 & 26 & 42 & 18 \\ 10 & 18 & 1 & 14 & 22 \\ 33 & 24 & 20 & 16 & 37 \end{bmatrix}$$

a. *(2 marks)* True or False: There exists an elementary matrix E such that $EM = A$. Justify.

b. *(3 marks)* If possible show that M and A are row-equivalent. Justify.

[1] From a John Abbott Final Examination

Question 7. *(5 marks)* Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$E_3(E_1X^T + \mathrm{trace}(E_2)A) = A$$

solve for X, if possible.

Bonus Question. *(5 marks)*
Prove: A square matrix A is invertible if and only if $AB = AC$ implies $B = C$.