Dawson College: **Linear Algebra**: 201-NYC-05-S04: Winter 2017

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Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$A = \begin{bmatrix} 0 & 2 & -5 & -1 \\ 2 & -6 & 0 & 3 \\ 3 & 0 & 0 & -2 \\ 5 & -4 & -5 & 0 \end{bmatrix} \underset{\substack{R_1 \leftrightarrow R_2 \\ 2R_3 \to R_3 \\ 2R_4 \to R_4}}{\sim} \begin{bmatrix} 2 & -6 & 0 & 3 \\ 0 & 2 & -5 & -1 \\ 6 & 0 & 0 & -4 \\ 10 & -8 & -10 & 0 \end{bmatrix} \underset{\substack{-3R_1+R_3 \to R_3 \\ -5R_1+R_4 \to R_4}}{\sim} \begin{bmatrix} 2 & -6 & 0 & 3 \\ 0 & 2 & -5 & -1 \\ 0 & 18 & 0 & -13 \\ 0 & 22 & -10 & -15 \end{bmatrix}$$

a. *(5 marks)* Find the reduced row echelon form of the matrix A.

b. *(2 marks)* Suppose that A is the augmented matrix of a linear system. Write the corresponding system of linear equations. Use part a to find the solution of the system.

c. *(2 marks)* Suppose that A is the coefficient matrix of a homogeneous linear system. Write the corresponding system of linear equations. Use part a. to find the solution of the system.

d. *(1 mark)* Suppose that A is the coefficient matrix of a homogeneous linear system. Use part c. to find a solution to the system when $x_1 = -1$.

e. *(2 marks)* Express A as a product of elementary matrices, if possible. Justify.

f. *(1 mark)* **True** or **False**: The reduced row echelon form of any matrix is unique. *(do not justify!)*

$$\underset{\substack{-9R_2+R_3 \to R_3 \\ -11R_2+R_4 \to R_4}}{\sim} \begin{bmatrix} 2 & -6 & 0 & 3 \\ 0 & 2 & -5 & -1 \\ 0 & 0 & 45 & -4 \\ 0 & 0 & 45 & -4 \end{bmatrix} \underset{\substack{9R_2 \to R_2 \\ -R_3+R_4 \to R_4}}{\sim} \begin{bmatrix} 2 & -6 & 0 & 3 \\ 0 & 18 & -45 & -9 \\ 0 & 0 & 45 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_3+R_2 \to R_2}{\sim} \begin{bmatrix} 2 & -6 & 0 & 3 \\ 0 & 18 & 0 & -13 \\ 0 & 0 & 45 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{3R_1 \to R_1}{\sim} \begin{bmatrix} 6 & -18 & 0 & 9 \\ 0 & 18 & 0 & -13 \\ 0 & 0 & 45 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 6 & 0 & 0 & -4 \\ 0 & 18 & 0 & -13 \\ 0 & 0 & 45 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{\frac{1}{6}R_1 \to R_1 \\ \frac{1}{18}R_2 \to R_2 \\ \frac{1}{45}R_3 \to R_3}}{\sim} \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & -13/18 \\ 0 & 0 & 1 & -4/45 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b)
$$\begin{aligned} x_2 - 5x_3 &= -1 \\ 2x_1 - 6x_2 &= 3 \\ 3x_1 &= -2 \\ 5x_1 - 4x_2 - 5x_3 &= 0 \end{aligned}$$
and its solution is $(x_1, x_2, x_3) = \left(\frac{-2}{3}, \frac{-13}{18}, \frac{-4}{45}\right)$

c)
$$\begin{aligned} x_2 - 5x_3 - x_4 &= 0 \\ 2x_1 - 6x_2 + 3x_4 &= 0 \\ 3x_1 - 2x_4 &= 0 \\ 5x_1 - 4x_2 - 5x_3 &= 0 \end{aligned}$$
and its solution set is obtained by letting $x_4 = t$, $t \in \mathbb{R}$ and $x_1 = \frac{2}{3}t$, $x_2 = \frac{13}{18}t$, $x_3 = \frac{4}{45}t$

$\therefore (x_1, x_2, x_3, x_4) = \left(\frac{2}{3}t, \frac{13}{18}t, \frac{4}{45}t, t\right)$

d) if $x_1 = -1$ then $t = -\frac{3}{2}$ and the solution is $(x_1, x_2, x_3, x_4) = \left(-1, \frac{-13}{12}, \frac{-2}{15}, \frac{-3}{2}\right)$

e) By the equivalence thm it is not possible since the RREF of A is not I.

Question 2. Consider the matrices:

$$A = [a_{ij}]_{3\times3},\quad B = [b_{ij}]_{2\times3},\quad C = [c_{ij}]_{3\times2},\quad D = [d_{ij}]_{2\times2},\quad E = [e_{ij}]_{3\times6}$$

where $a_{ij} = i - j$, $b_{ij} = (-1)^i 2 + (-1)^j 3$, $c_{ij} = i + j$, $d_{ij} = (ij)^2$, $e_{ij} = i + j$. Evaluate the following if possible, justify.

a. *(2 marks)* AB $= A_{3\times3} B_{2\times3}$ is not defined since the number of columns of A is not equal to the number of rows of B

b. *(2 marks)* $B^T E$ $= B^T_{3\times2} E_{3\times6}$ is not defined, same reason at part a.

Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix},\ D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix},\ E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Evaluate the following if possible, justify.

a. *(2 marks)* $\text{trace}(C)\,AB - \text{trace}(D)B$

Not possible since AB is a 3x3 matrix and so is tr(C)AB and tr(D)B is a 2x2. In order to add matrices their dimension must be equal.

b. *(2 marks)* $(AB - 2I)A$ $= ABA - 2IA = ABA - 2A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix} A - 2A$

$$= \begin{bmatrix} -4 & -1 & 4 \\ 0 & -11 & 8 \\ 5 & -7 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix} - 2A$$

$$= \begin{bmatrix} -3 & -14 \\ -25 & -30 \\ -15 & -5 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & -18 \\ -31 & -34 \\ -17 & -2 \end{bmatrix}$$

Question 3. Given A and B, $n \times n$ matrices.

a. *(2 marks)* Prove or disprove: If $AB = 0$ then $A = 0$ or $B = 0$.

Disprove: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

but $A \neq 0$

and $B \neq 0$.

b. *(2 marks)* Prove or disprove: If A is invertible and $AB = 0$ then $B = 0$. Prove:

Premise:

- A is invertible, $AA^{-1} = A^{-1}A = I$
- $AB = 0$

Conclusion:

$B = 0$.

$$AB = 0$$
$$A^{-1}AB = A^{-1}0$$
$$IB = 0$$
$$B = 0 \quad ■$$

Question 4. Let A and B be $n \times n$ matrices.

a. *(2 marks)* Verify that $A(I + BA) = (I + AB)A$ and that $(I + BA)B = B(I + AB)$.

$$\begin{aligned} LHS &= A(I+BA) \\ &= AI + ABA \\ &= A + ABA \\ &= IA + ABA \\ &= (I+AB)A \\ &= RHS \end{aligned}$$

$$\begin{aligned} LHS &= (I+BA)B \\ &= IB + BAB \\ &= B + BAB \\ &= BI + BAB \\ &= B(I+AB) \\ &= RHS \end{aligned}$$

Prove:

b. *(3 marks)* If $I + AB$ is inverrible, verify that $I + BA$ is also invertible and that $(I + BA)^{-1} = I - B(I + AB)^{-1}A$.

Premise:

$I + AB$ is invertible, $(I+AB)(I+AB)^{-1} = I = (I+AB)^{-1}(I+AB)$

Conclusion:

$I + BA$ is invertible and its inverse is $I - B(I+AB)^{-1}A$.

Need to verify that ① $(I+BA)(I - B(I+AB)^{-1}A) = I$ and ② $(I - B(I+AB)^{-1}A)(I+BA) = I$

① $LHS = (I+BA)(I - B(I+AB)^{-1}A) = I \cdot I - I \cdot B(I+AB)^{-1}A + BA \cdot I - BAB(I+AB)^{-1}A)$

$= I + BA - (I + BA)B(I+AB)^{-1}A$ by part a.

$= I + BA - B(I+AB)(I+AB)^{-1}A = I + BA - BIA = I = RHS$

② $LHS = (I - B(I+AB)^{-1}A)(I+BA) = I \cdot I + IBA - B(I+AB)^{-1}AI - B(I+AB)^{-1}ABA$

$= I + BA - B(I+AB)^{-1}A(I+BA)$ by part a

$= I + BA - B(I+AB)^{-1}(I+AB)A = I + BA - BIA = I = RHS$ ■

Question 5.[1] *(5 marks)* The number of leading 1's in a row echelon form of A is called the *rank* of A. Given the following matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & b \\ 0 & 3 & a & b \end{bmatrix} \underset{\substack{-R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & b-1 \\ 0 & 3 & a & b \end{bmatrix} \underset{\substack{-2R_2+R_3\to R_3 \\ -3R_2+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & b-3 \\ 0 & 0 & a+3 & b-3 \end{bmatrix}$$

$$\underset{R_3\leftrightarrow R_4}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & a+3 & b-3 \\ 0 & 0 & 0 & b-3 \end{bmatrix}$$

a. Under what conditions on a and b is the rank of A equal to 4.

b. Under what conditions on a and b is the rank of A equal to 3.

c. Under what conditions on a and b is the rank of A equal to 2.

a) The rank of A is equal to 4. if there are 4 leading entries. $\therefore$ $a+3 \neq 0$ and $b-3 \neq 0$

b) rank(A) = 3 if there are 3 leading entries. Two possibilities:

① $b-3=0$ and $a+3 \neq 0$

② $b-3 \neq 0$ and $a+3=0$

c) rank(A) = 2 if there are 2 leading entries. $\therefore$ $a+3=0$ and $b-3=0$.

Question 6. Given a 5×5 *Magic Square (matrix)*

$$M = \begin{bmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{bmatrix} \text{ and } A = \begin{bmatrix} 4 & 12 & 25 & 8 & 16 \\ 11 & 24 & 7 & 20 & 3 \\ 34 & 10 & 26 & 42 & 18 \\ 10 & 18 & 1 & 14 & 22 \\ 33 & 24 & 20 & 16 & 37 \end{bmatrix}$$

a. *(2 marks)* True or False: There exists an elementary matrix E such that $EM = A$. Justify.

b. *(3 marks)* If possible show that M and A are row-equivalent. Justify.

a) False,

Notice that A and M differ by the following elem. row. op. $2R_3 \to R_3$, $R_1 \leftrightarrow R_2$, $R_4 + R_5 \to R_5$. Multiplying on the left by an elementary matrix is equivalent to performing an elementary operation. Hence not possible

b) M and A are row equivalent since $M \underset{\substack{R_1\leftrightarrow R_2 \\ 2R_3\to R_3 \\ R_4+R_5\to R_5}}{\sim} A$.

[1] From a John Abbott Final Examination

Question 7. *(5 marks)* Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$E_3(E_1X^T + \text{trace}(E_2)A) = A$$

solve for X, if possible.

note:

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_3^{-1}E_3(E_1X^T - A) = E_3^{-1}A$$

$$I(E_1X^T - A) = E_3^{-1}A$$

$$E_1X^T - A = E_3^{-1}A$$

$$E_1X^T = E_3^{-1}A + A$$

$$E_1^{-1}E_1X^T = E_1^{-1}(E_3^{-1}A + A)$$

$$IX^T = E_1^{-1}(E_3^{-1}A + A)$$

$$X = (E_1^{-1}(E_3^{-1}A + A))^T$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \right)\right)^T$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} \right)^T$$

$$= \left(\begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ -4 & -4 & -4 & -4 \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} 4 & 4 & -4 \\ 4 & 4 & -4 \\ 4 & 4 & -4 \\ 4 & 4 & -4 \end{bmatrix}$$

Bonus Question. *(5 marks)*
Prove: A square matrix A is invertible if and only if $AB = AC$ implies $B = C$.

[$\Rightarrow$]

Premise:

A is invertible $\Rightarrow AA^{-1} = I = A^{-1}A$

Conclusion:

$AB = AC$ implies $B = C$

$$AB = AC$$
$$A^{-1}AB = A^{-1}AC$$
$$IB = IC$$
$$B = C$$

[$\Leftarrow$]

Premise:

$AB = AC$ implies $B = C$

Conclusion: A is invertible

Given the homogeneous system $Ax = 0$

$$Ax = A0$$
$$x = 0 \text{ by the premise}$$

Hence $x = 0$ is a solution, if x is any solution then it must be the trivial solution.

$\therefore$ by the equivalence theorem since $Ax = 0$ has only the trivial solution then A is invertible.