Dawson College: **Linear Algebra**: 201-NYC-05-S04: Winter 2017

Name: Y. Lamontagne

Test 2

This test is graded out of 45 marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531XG or Sharp EL-531X calculator. Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

Question 1. Given

$$A = \begin{bmatrix} 2 & 1 & 4 & 4 \\ -3 & 0 & -3 & -4 \\ 3 & 2 & 0 & 0 \\ 4 & 0 & 5 & 0 \end{bmatrix}.$$

$$|A| \overset{R_1+R_2\to R_2}{=} \begin{vmatrix} 2 & 1 & 4 & 4 \\ -1 & 1 & 1 & 0 \\ 3 & 2 & 0 & 0 \\ 4 & 0 & 5 & 0 \end{vmatrix} = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

$$= 4(-1)^{1+4}\begin{vmatrix} -1 & 1 & 1 \\ 3 & 2 & 0 \\ 4 & 0 & 5 \end{vmatrix}$$

$$\overset{\substack{3R_1+R_2\to R_2 \\ 4R_1+R_3\to R_3}}{=} 4\begin{vmatrix} -1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 4 & 9 \end{vmatrix}$$

$$= 4(a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31})$$

$$= 4(-1\,C_{11})$$

$$= 4\left(-1(-1)^{1+1}\begin{vmatrix} 5 & 3 \\ 4 & 9 \end{vmatrix}\right)$$

$$= 4(-(45-12))$$

$$= 4(-33) = 132$$

a. *(5 marks)* Evaluate $\det(A)$.

b. *(5 marks)* If M is a 4×4 matrix such that $\det(M) = 3$ then evaluate $\det(\det(5A^T)\operatorname{adj}(MA^{-1}))$. Justify!

b) $\det(\det(5A^T)\,\mathrm{adj}(MA^{-1}))$

$$= (\det 5A^T)^4 \det(\mathrm{adj}(MA^{-1}))$$

$$= (5^4 \det A^T)^4 (\det(MA^{-1}))^{4-1}$$

$$= (5^4 \det A)^4 (\det M \det A^{-1})^{4-1}$$

$$= 5^{16}(132)^4(3)^3\left(\frac{1}{\det A}\right)^3$$

$$= 5^{16}(132)^4\, 3^3\left(\frac{1}{132}\right)^3$$

$$= 5^{16}\cdot 132\cdot 3^3$$

Question 2. *(3 marks)* Prove: There does not exist $n \times n$ invertible matrices A and B where A is symmetric, B is skew symmetric, n is odd such that AB is symmetric.

Proof by contradiction: Suppose that there exists such an A & B.

$$(AB)^T = AB$$
$$B^T A^T = AB$$
$$B^T A = AB \quad \text{since } A \text{ is symmetric (i.e. } A^T = A)$$
$$-BA = AB \quad \text{since } B \text{ is skew-symmetric}$$

$$\det(-BA) = \det(AB)$$
$$(-1)^n \cdot \det B \det A = \det A \det B$$
$$-\det A \det B = \det A \det B$$
$$\neq 0 \quad \neq 0 \qquad \neq 0 \quad \neq 0$$

since n is odd

$\therefore$ there does not exists such an A & B.

Question 3. *(3 marks)* Prove or disprove: If $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution then $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have only the trivial solution.

Prove: premise: $(AB)x = 0$ has only the trivial solution
conclusion: $Ax = 0$ and $Bx = 0$ only have the trivial solution

Since $(AB)x = 0$ only has the trivial solution, AB is invertible by the equivalence theorem. If AB is invertible then so is A & B. Hence by the equivalence theorem $Ax = 0$ and $Bx = 0$ only have the trivial solution.

Question 4. *(3 marks)* Prove or disprove: There does not exist two unit vectors $\vec{u}$, $\vec{v} \in \mathbb{R}^n$ such that $\vec{u} \cdot \vec{v} = -2$.

Prove: Proof by contradiction
Suppose there exists two unit vectors $\vec{u}$ and $\vec{v}$ such that $\vec{u} \cdot \vec{v} = -2$.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$
$$-2 = 1 \cdot 1 \cos\theta$$
$$-2 = \cos\theta$$

since the range of $\cos\theta$ is $[-1, 1]$.

Question 4. Given the plane $4x+3y+2z=12$

a. *(2 marks)* Sketch the given plane using the x, y and z-intercepts.

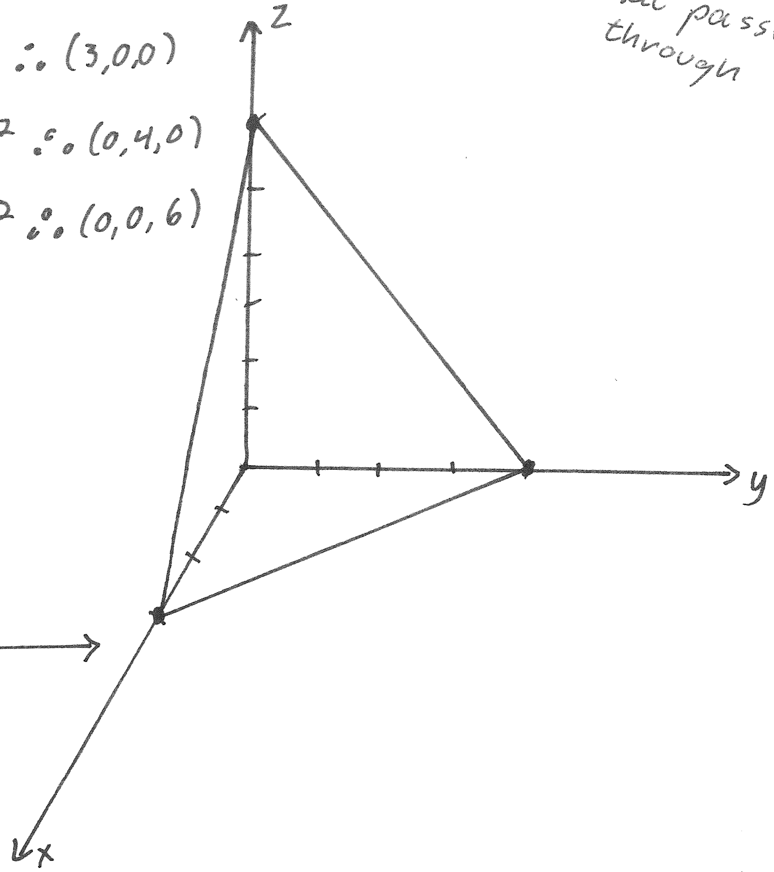
b. *(5 marks)* Using projections find the shortest distance between the y-intercept and the line that passes through the x, z-intercepts.

c. *(2 marks)* Find the equation of the line which passes through the y-intercept and the closest point on the line that passes through the x, z-intercepts to the y-intercept.

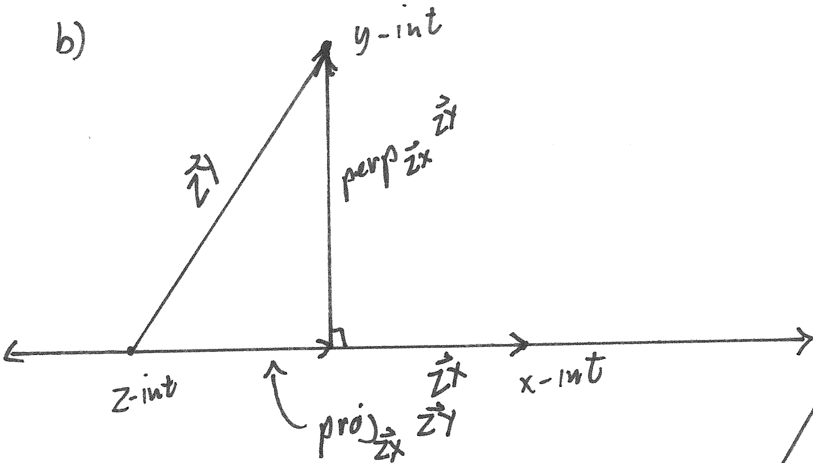
a) x-int: let $y=z=0$ $\quad 4x+3(0)+2(0)=12$, $x=3$ $\quad \therefore (3,0,0)$

y-int: let $x=z=0$ $\quad 4(0)+3y+2(0)=12$, $y=4$ $\quad \therefore (0,4,0)$

z-int: let $x=y=0$ $\quad 4(0)+3(0)+2z=12$, $y=6$ $\quad \therefore (0,0,6)$



b)



$\vec{ZX} = X - Z = (3,0,0)-(0,0,6)$
$= (3,0,-6)$

$\vec{ZY} = Y - Z = (0,4,0) - (0,0,6)$
$= (0,4,-6)$

$perp_{\vec{ZX}}\vec{ZY} = \vec{ZY} - proj_{\vec{ZX}}\vec{ZY}$

$= (0,4,-6) - \dfrac{\vec{ZY}\cdot\vec{ZX}}{\vec{ZX}\cdot\vec{ZX}}\vec{ZX}$

$= (0,4,-6) - \dfrac{36}{9+36}(3,0,-6)$

$= (0,4,-6) - \dfrac{36}{45}(3,0,-6)$

$= \left(\dfrac{-108}{45}, 4, -6+\dfrac{6(36)}{45}\right)$

$= \left(\dfrac{-108}{45}, 4, \dfrac{-54}{45}\right) = \left(-\dfrac{12}{5}, 4, -\dfrac{6}{5}\right)$

$d = \| perp_{\vec{ZX}}\vec{ZY} \|$

$= \| \frac{1}{45}(-108, 4(45), -54) \|$

$= \frac{1}{45}\sqrt{(-108)^2+(180)^2+(-54)^2} = \frac{1}{45}\sqrt{46980} = \sqrt{\frac{1044}{45}} = \sqrt{\frac{116}{5}} = 2\sqrt{\frac{29}{5}}$

c) $\vec{x} = p + t\vec{d}$

$= (0,4,0) + t\, perp_{\vec{ZX}}\vec{ZY}$

$= (0,4,0) + t\left(\frac{-108}{45}, 4, \frac{-54}{45}\right)$

$= (0,4,0) + t\left(-\frac{12}{5}, 4, -\frac{6}{5}\right)$

Question 5. *(4 marks)* Using elementary operations show that

$$-sr\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} sb+2d & rsa+2rc \\ d & rc \end{vmatrix}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} sb+2d & rsa+2rc \\ d & rc \end{bmatrix}$

(op that change det) det[orig. mat]
= det [new mat.]

$\sim -2R_2+R_1 \to R_1 \begin{bmatrix} sb & rsa \\ d & rc \end{bmatrix}$

$(\frac{1}{s}\frac{1}{r}(-1))\ B = A$

$B = -sr\ A$

$\sim \frac{1}{s}R_1 \to R_1 \begin{bmatrix} b & ra \\ d & rc \end{bmatrix}$ if $s \neq 0$

$\sim \frac{1}{r}C_2 \to C_2 \begin{bmatrix} b & a \\ d & c \end{bmatrix}$ if $r \neq 0$

$\sim C_1 \leftrightarrow C_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

If $s=0$ then LHS $=0$ RHS $= \begin{vmatrix} 2d & 2rc \\ d & rc \end{vmatrix} = 0$

If $r=0$ then LHS $=0$ RHS $= \begin{vmatrix} sb & 0 \\ d & 0 \end{vmatrix} = 0$

Question 6. *(3 marks)* Prove: If $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $||\vec{u}|| = \sqrt{2}$ and the angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{4}$ then $||\vec{u}+\vec{v}||^2 = ||\vec{v}||^2 + ||\vec{v}|| + 2$.

$$\begin{aligned}
||\vec{u}+\vec{v}||^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\
&= ||\vec{u}||^2 + 2\vec{u}\cdot\vec{v} + ||\vec{v}||^2 \\
&= (\sqrt{2})^2 + 2||\vec{u}||||\vec{v}||\cos\frac{\pi}{4} + ||\vec{v}||^2 \\
&= 2 + 2\sqrt{2}||\vec{v}||\frac{1}{\sqrt{2}} + ||\vec{v}||^2 \\
&= ||\vec{v}||^2 + 2||\vec{v}|| + 2.
\end{aligned}$$

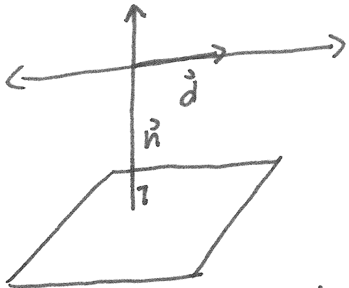
Question 7. Given the plane $x+y+z=0$ and the line $(x,\ y,\ z)=(1+t, 2+2t, 3+3t)$ where $t\in\mathbb{R}$.

a. *(2 marks)* Determine whether the line is perpendicular to the plane, parallel or neither. Justify!

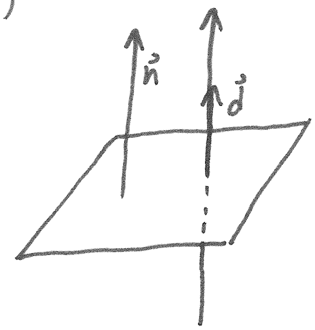
b. *(3 marks)* Find the point of intersection between the line and the plane if it exists.

c. *(3 marks)* Find the smallest angle between the line and the plane.

a)



b)



If $\vec{n} \parallel \vec{d}$ then the line and plane are $\perp$.
$\nexists K$ s.t. $\vec{n} = K\vec{d}$.
Hence the line and plane are not perpendicular

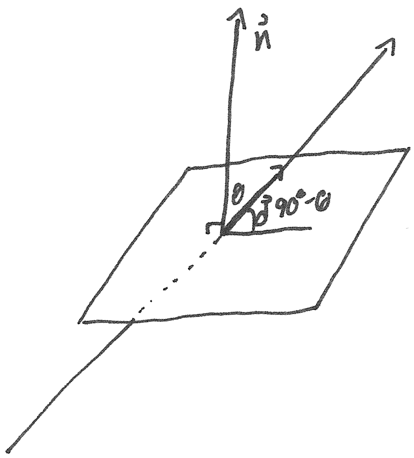
If $\vec{n} \perp \vec{d}$ then the line and plane are $\parallel$.
$\vec{n}\cdot\vec{d} = (1,1,1)\cdot(1,2,3) \neq 0$
$\therefore$ $\vec{n}$ and $\vec{d}$ are not perpendicular
$\therefore$ Line and plane are not $\parallel$.

b) $\mathcal{L}$: $x = 1+t$, $y = 2+2t$, $z = 3+3t$ sub into $x+y+z=0$

$(1+t)+(2+2t)+(3+3t) = 0$
$6+6t = 0$
$t = -1$

Hence intersection when $t=-1$
$\therefore (x,y,z) = (1-1, 2+2(-1), 3+3(-1))$
$= (0,0,0)$

c)



$\vec{n}\cdot\vec{d} = \|\vec{n}\|\|\vec{d}\|\cos\theta$
$(1,1,1)\cdot(1,2,3) = \|(1,1,1)\|\|(1,2,3)\|\cos\theta$
$1+2+3 = \sqrt{1^2+1^2+1^2}\sqrt{1^2+2^2+3^2}\ \cos\theta$
$6 = \sqrt{3}\sqrt{14}\cos\theta$
$\dfrac{6}{\sqrt{3}\sqrt{14}} = \cos\theta$
$\theta = \arccos\left(\dfrac{6}{\sqrt{3}\sqrt{14}}\right)$
$= 22.20°$

$\therefore$ the angle between the plane and the line is $90-22.20° = 67.8°$

Bonus Question. *(5 marks)*
Prove: If A and B are two matrices such that $A+B=AB$ then A and B commute.

Given $A+B = AB$ we get $A = AB - B = (A-I)B$ and $AB - A - B = 0$

$B = AB - A = A(B-I)$

So $BA = A\left[(B-I)(A-I)\right]B$

$= AIB$

$= AB$

note that: $(A-I)(B-I)$

$= \underbrace{AB - A - B} + I$

$= 0 + I$

$= I$

Hence $(A-I)$ and $(B-I)$ are inverses of each other

Solution by Bogdan.