Dawson College: **Linear Algebra**: 201-NYC-05-S04: Winter 2017

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Test 3

This test is graded out of 43 marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531XG or Sharp EL-531X calculator. Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

Question 1. Given:

$$\begin{aligned} \mathscr{L}_1 &: (x,y,z) = \overbrace{(1,0,1)}^{P_1} + t_1\overbrace{(-2,-1,0)}^{\vec{d}_1} \\ \mathscr{L}_2 &: (x,y,z) = \underbrace{(-2,-1,2)}_{P_2} + t_2\underbrace{(1,0,1)}_{\vec{d}_2} \text{ where } t_1,\ t_2 \in \mathbb{R}. \end{aligned}$$

a. *(3 marks)* Determine whether the two lines intersect, are parallel or are skew lines.

b. *(5 marks)* Find a point on each line which is closest to the other line, are those points unique?

c. *(3 marks)* Find the equation of a plane that contains the line which passes through the point on each line which is closest to the other line. Is the plane unique?

a) $\mathcal{L}_1$ and $\mathcal{L}_2$ are not parallel since $\vec{d}_1 \parallel \vec{d}_2$. Do $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect?

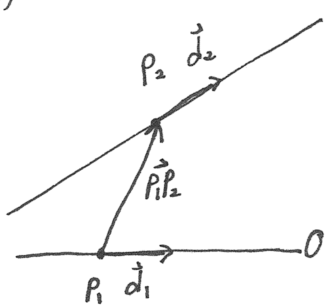
① $1 - 2t_1 = -2 + t_2$

② $-t_1 = -1 \Rightarrow t_1 = 1$ sub into ① $1 - 2(1) = -2 + t_2$, $1 = t_2$ check consistency: $1 \overset{?}{=} 2+1$, $1 \overset{?}{=} 3$

③ $1 = 2 + t_2$

Not consistent. ∴ no intersection ∴ $\mathcal{L}_1$ and $\mathcal{L}_2$ are skew lines.

b)



$\vec{P_1P_2} = P_2 - P_1$

$= (-2+t_2, -1, 2+t_2) - (1-2t_1, -t_1, 1)$

$= (-3 + t_2 + 2t_1, -1 + t_1, 1 + t_2)$

Lets determine when $\vec{P_1P_2} \perp \vec{d}_1$ and $\vec{P_1P_2} \perp \vec{d}_2$

$0 = \vec{P_1P_2} \cdot \vec{d}_1 = -2(-3 + t_2 + 2t_1) - (-1 + t_1)$

$0 = 6 - 2t_2 - 4t_1 + 1 - t_1$

① $0 = 7 - 2t_2 - 5t_1$

$0 = \vec{P_1P_2} \cdot \vec{d}_2 = -3 + t_2 + 2t_1 + 1 + t_2$

$0 = -2 + 2t_2 + 2t_1$

$0 = -1 + t_2 + t_1$

② $1 - t_2 = t_1$

sub ② into ① $0 = 7 - 2t_2 - 5(1 - t_2)$

$0 = 2 + 3t_2$

$\frac{-2}{3} = t_2$

sub into ② $t_1 = 1 - \left(\frac{-2}{3}\right) = \frac{5}{3}$

∴ point on $\mathcal{L}_1$: $(1,0,1) + \frac{5}{3}(-2,-1,0) = \left(\frac{-7}{3}, \frac{-5}{3}, 1\right)$, on $\mathcal{L}_2$: $(-2,-1,2) - \frac{2}{3}(1,0,1) = \left(\frac{-8}{3}, -1, \frac{4}{3}\right)$

Those points are unique since there exists only one vector $\perp$ to $\vec{d}_1$ and $\vec{d}_2$.

c) $(x,y,z) = \left(\frac{-7}{3}, \frac{-5}{3}, 1\right) + t\vec{d}_1 + s\vec{d}_2$ where $\vec{d}_1 = \left(\frac{-8}{3}, -1, \frac{4}{3}\right) - \left(\frac{-7}{3}, \frac{-5}{3}, 1\right) = \left(\frac{-1}{3}, \frac{2}{3}, \frac{1}{3}\right)$

and $\vec{d}_2$ = any vector not $\parallel$ to $\vec{d}_1$
$= (1,0,0)$

Hence the plane is not unique as the second direction vector has ∞ many possibilities

Question 2. *(3 marks)* Given a parallepiped (le) generated by the vectors $\vec{u} = (1,-1,-1), \vec{v}$ and $\vec{w} \in \mathbb{R}^3$ with volume 2017 and $\frac{\vec{v}\times\vec{w}}{\|\vec{v}\times\vec{w}\|} = \frac{1}{\sqrt{14}}(1,2,3)$. Find the area of the base defined by $\vec{v}$ and $\vec{w}$.

$|\vec{u}\cdot(\vec{v}\times\vec{w})| = 2017$ where $\vec{v}\times\vec{w} = \frac{\|\vec{u}\times\vec{v}\|}{\sqrt{14}}(1,2,3)$

$\left|(1,1,1)\cdot \frac{\|\vec{u}\times\vec{v}\|}{\sqrt{14}}(1,2,3)\right| = 2017$

$\frac{\|\vec{u}\times\vec{v}\|}{\sqrt{14}}\,|(1,-1,-1)\cdot(1,2,3)| = 2017$

$\frac{\|\vec{u}\times\vec{v}\|}{\sqrt{14}}|(1-2-3)| = 2017$

$\|\vec{u}\times\vec{v}\| = \frac{2017\cdot\sqrt{14}}{4}$

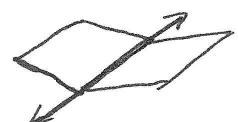
$\therefore$ the area of the base (parallelogram) is $\frac{2017\sqrt{14}}{4}$

Question 3. Given two planes:

$$\begin{aligned}\mathscr{P}_1 &: \quad x+z = 1\\ \mathscr{P}_2 &: \quad y+z = 1\end{aligned}$$

a) $(0,0,1)$

b) The normal of $\mathscr{P}_1$ and $\mathscr{P}_2$ are not parallel so the two planes will intersect at a line



a. *(1 mark)* Give a point of intersection of the two planes, by inspection.

b. *(1 mark)* Give a geometrical argument to explain why the intersection of the two planes is a line.

c. *(2 marks)* Find the direction vector for the intersection of the two planes without solving for the solution set. Justify.

d. (1 mark) Find the sol. set of the system by only using part a) and part c)

c) $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \left(\begin{vmatrix}0&1\\1&1\end{vmatrix}, -\begin{vmatrix}1&1\\0&1\end{vmatrix}, \begin{vmatrix}1&0\\0&1\end{vmatrix}\right) = (-1,-1,1)$

$\begin{matrix}1 & 0\\ 0 & 1\\ 1 & 1\end{matrix}$

Since the line of intersection is orthogonal to both lines.

d) $(x,y,z) = (0,0,1) + t(-1,-1,1)$ where $t \in \mathbb{R}$

Question 4. The number of leading 1's in a row echelon form of A is called the *rank* of A. Let $V = \{M \mid M \in \mathscr{M}_{2\times 2} \text{ and } \operatorname{rank}(M) = 2\}$ with the following vector addition and scalar multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix} \text{ and } k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^k & b^k \\ c^k & d^k \end{bmatrix}$$

a. *(3 marks)* Is the zero vector an element of V? Justify.

b. *(3 marks)* Determine whether the following axiom holds: $(rs)M = r(sM)$ where $r, s \in \mathbb{R}$ and $M \in V$

a) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ and $\vec{0} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$M + \vec{0} = M$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$ae = a \Rightarrow e = 1$
$bf = b \Rightarrow f = 1$
$cg = c \Rightarrow g = 1$
$dh = d \Rightarrow h = 1$

$\vec{0} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin V$

since $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim_{-R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has rank 1.

b) Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ and $r, s \in \mathbb{R}$

$LHS = (rs)M$
$= (rs)\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^{rs} & b^{rs} \\ c^{rs} & d^{rs} \end{bmatrix}$

$RHS = r(sM)$
$= r\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$
$= r\begin{bmatrix} a^s & b^s \\ c^s & d^s \end{bmatrix}$
$= \begin{bmatrix} (a^s)^r & (b^s)^r \\ (c^s)^r & (d^s)^r \end{bmatrix}$
$= \begin{bmatrix} a^{rs} & b^{rs} \\ c^{rs} & d^{rs} \end{bmatrix}$

axiom holds since LHS=RHS

Question 5. *(3 marks)* Determine whether $W = \{a_0 + a_1x + a_2x^2 \mid a_0a_1a_2 = 0\}$ is a subspace of P_2.

Let $p(x) = 1 + 0x + 1\cdot x^2 \in W$ since $1\cdot 0\cdot 1 = 0$
$q(x) = 0 + 1\cdot x + 0x^2 \in W$ since $0\cdot 1\cdot 0 = 0$

$p(x) + q(x) = 1 + 1\cdot x + 1\cdot x^2 \notin W$ since $1\cdot 1\cdot 1 \neq 0$

$\therefore$ not a subspace of P_2.

Question 6. Given $\vec{u} = (1,-1,1), \vec{v} = (-1,1,\lambda), \vec{w} = (1,2,3) \in \mathbb{R}^3$ and the set $S = \{\vec{u},\ \vec{v},\ (\vec{u}\cdot\vec{v})\vec{w}\}$.

a. *(2 marks)* Find the value(s), if any, of λ for which S spans $\mathbb{R}^3$

b. *(2 marks)* Find the value(s), if any, of λ for which S spans a plane.

c. *(2 marks)* Find the value(s), if any, of λ for which S spans a line.

a) $\vec{0} = c_1\vec{u} + c_2\vec{v} + c_3(\vec{u}\cdot\vec{v})\vec{w}$

$(0,0,0) = c_1(1,-1,1) + c_2(-1,1,\lambda) + c_3(\lambda-2)(1,2,3)$

$$\underbrace{\begin{bmatrix} 1 & -1 & (\lambda-2) \\ -1 & 1 & (\lambda-2)2 \\ 1 & \lambda & (\lambda-2)3 \end{bmatrix}}_{A}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has only the trivial sol. if $|A| \neq 0$ by the equivalence thm

$$|A| = (\lambda-2)\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \\ 1 & \lambda & 3 \end{vmatrix}$$

$$\underset{R_2+R_1\to R_1}{=} (\lambda-2)\begin{vmatrix} 0 & 0 & 3 \\ -1 & 1 & 2 \\ 1 & \lambda & 3 \end{vmatrix} = (\lambda-2)3\begin{vmatrix} -1 & 1 \\ 1 & \lambda \end{vmatrix} = (\lambda-2)3(-\lambda-1) \neq 0$$

$\therefore \lambda \neq 2$ and $\lambda \neq -1$

$\vec{u}\cdot\vec{v} = -1-1+\lambda = -2+\lambda$

b) If $\lambda = 2$ then since $(\vec{u}\cdot\vec{v})\vec{w} = \vec{0}$

$\text{span}(S) = \text{span}(\{(1,-1,1),(-1,1,2)\})$
$= \{c_1(1,-1,1) + c_2(-1,1,2) \mid c_i \in \mathbb{R}\}$

is a plane

If $\lambda = -1$ then since $\vec{u} = -\vec{v}$

$\text{span}(S) = \text{span}(\{(1,-1,1),(1,2,3)\})$
$= \{c_1(1,-1,1) + c_2(1,2,3) \mid c_i \in \mathbb{R}\}$

is a plane

c) No such value exist. all posibilities exhausted in part a, b.

Question 7. Given $W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a+2b+3c+4d = 0 \text{ and } b+c+d = 0\right\}$ a subspace of $\mathscr{M}_{2\times2}$.

a. *(3 marks)* Find a basis B for W.

b. *(1 mark)* State the $\dim(W)$ and $\dim(\mathscr{M}_{2\times2})$.

a) Find all a, b, c, d s.t. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$

$$\begin{matrix} a+2b+3c+4d=0 \\ b+c+d=0 \end{matrix} \equiv \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \underset{\sim}{\xrightarrow{-2R_2+R_1\to R_1}} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

let $c = s$, $d = t$, $s,t \in \mathbb{R}$
$a = -s-2t$
$b = -s-t$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -s-2t & -s-t \\ s & t \end{bmatrix} = s\underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}}_{M_1} + t\underbrace{\begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}}_{M_2}$$

So $W = \text{span}(\{M_1, M_2\})$ and $\{M_1, M_2\}$ is linearly independent since M_1 is not a multiple of M_2. $\therefore \{M_1, M_2\}$ is a basis of W.

b) $\dim(W) = 2$
$\dim(M_{2\times2}) = 4$

Bonus Question. *from Wikipedia (3 marks)*
Given $R = \{x \mid x \notin x\}$. Discuss the following statement: $R \in R \longleftrightarrow R \notin R$.