Dawson College: **Linear Algebra**: 201-NYC-05-S05: Winter 2017

Name: ____________________

Test 1

This test is graded out of 44 marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531XG or Sharp EL-531X calculator. Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

Question 1. Given

$$A = \begin{bmatrix} 0 & 2 & 0 & 8 \\ 0 & -6 & 0 & 3 \\ 3 & 7 & 0 & -2 \\ 5 & 10 & 3 & 7 \end{bmatrix}$$

a. *(5 marks)* Find the reduced row echelon form of the matrix A.

b. *(1 mark)* Suppose that A is the augmented matrix of a linear system. Write the corresponding system of linear equations. Use part a. to find the solution of the system.

c. *(2 marks)* Suppose that A is the coefficient matrix of a homogeneous linear system. Write the corresponding system of linear equations. Use part a. to find the solution of the system.

d. *(2 marks)* **True** or **False**: A is expressible as a product of elementary matrices. Justify.

f. *(1 mark)* **True** or **False**: The row echelon form of A is unique. *(do not justify!)*

g. *(1 mark)* **True** or **False**: A is invertible. *(do not justify!)*

Question 2. Consider the matrices:

$$A = [a_{ij}]_{3\times 3}, \quad B = [b_{ij}]_{2\times 3}, \quad C = [c_{ij}]_{3\times 2}, \quad D = [d_{ij}]_{2\times 2}, \quad E = [e_{ij}]_{3\times 6}$$

where $a_{ij} = i - j, \quad b_{ij} = (-1)^i 2 + (-1)^j 3, \quad c_{ij} = i + j, \quad d_{ij} = (ij)^2, \quad e_{ij} = i + j$. Evaluate the following if possible, justify.

a. *(2 marks)* AB

b. *(2 marks)* AE

Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}, D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Evaluate the following if possible, justify.

a. *(2 marks)* $\text{trace}(C)BA - \text{trace}(D)EE^T$

b. *(2 marks)* $C(BA + 2I)$

Question 3. Let A, B and C denote $n \times n$ matrices.

a. *(2 marks)* Prove or disprove: If $AB = AC$ then $B = C$.

b. *(2 marks)* Prove or disprove: If A is invertible and $AB = AC$ then $B = C$.

Question 4. Let A and B denote $n \times n$ invertible matrices.

a. *(2 marks)* Show that $A^{-1} + B^{-1} = A^{-1}(A+B)B^{-1}$

b. *(3 marks)* If $A+B$ is also invertible, show that $A^{-1}+B^{-1}$ is invertible and find a formula for $(A^{-1}+B^{-1})^{-1}$.*(Hint: use part a.)*

Question 5. Given the 5×5 *Hilbert matrix* and its inverse

$$H = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}, \quad H^{-1} = \begin{bmatrix} 25 & -300 & 1050 & -1400 & 630 \\ -300 & 4800 & -18900 & 26880 & -12600 \\ 1050 & -18900 & 79380 & -117600 & 56700 \\ -1400 & 26880 & -117600 & 179200 & -88200 \\ 630 & -12600 & 56700 & -88200 & 44100 \end{bmatrix} \text{ and } A = \begin{bmatrix} \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\ 1 & \frac{2}{3} & \frac{1}{2} & \frac{2}{5} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

a. *(2 marks)* Show that H is row equivalent to A by finding two elementary matrices E_i such that $E_2E_1H = A$.

b. *(3 marks)* Find the inverse of A. *(Hint: use part a.)*

Question 6.[1] *(5 marks)* Let

$$A = \begin{bmatrix} 1 & 1 & a \\ 1 & a & a \\ a & a & a \\ a & a & a^2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a^2 - 2a \end{bmatrix}$$

a. For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have no solution?
b. For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have a unique solution?
c. For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions?

[1] From a John Abbot Final Examination

Question 7. *(5 marks)* Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$E_3\left(\text{trace}(E_1)X^T + E_2A\right) = A$$

solve for X, if possible.

Bonus Question. *(5 marks)*
Prove: If E is an elementary matrix then E^T is an elementary matrix.