Dawson College: **Linear Algebra**: 201-NYC-05-S05: Winter 2017

Name: Y. Lamontagne

Test 1

This test is graded out of 40 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Given

$$A = \begin{bmatrix} 0 & 2 & 0 & 8 \\ 0 & -6 & 0 & 3 \\ 3 & 7 & 0 & -2 \\ 5 & 10 & 3 & 7 \end{bmatrix} \sim_{R_1 \leftrightarrow R_4} \begin{bmatrix} 5 & 10 & 3 & 7 \\ 0 & -6 & 0 & 3 \\ 3 & 7 & 0 & -2 \\ 0 & 2 & 0 & 8 \end{bmatrix} \sim_{\substack{\frac{1}{3}R_2 \to R_2 \\ 5R_3 \to R_3 \\ \frac{1}{2}R_4 \to R_4}} \begin{bmatrix} 5 & 10 & 3 & 7 \\ 0 & -2 & 0 & 1 \\ 15 & 35 & 0 & -10 \\ 0 & 1 & 0 & 4 \end{bmatrix}$$

a. *(5 marks)* Find the reduced row echelon form of the matrix A.

b. *(2 marks)* Suppose that A is the augmented matrix of a linear system. Write the corresponding system of linear equations. Use the first part to find the solution of the system.

c. *(2 marks)* Suppose that A is the coefficient matrix of a homogeneous linear system. Write the corresponding system of linear equations. Use the first part to find the solution of the system.

d. Is it possible to *(2 marks)* Express A as a product of elementary matrices, Justify.

$$\sim_{\substack{R_2 \leftrightarrow R_4 \\ -3R_1 + R_3 \to R_3}} \begin{bmatrix} 5 & 10 & 3 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 5 & -9 & -31 \\ 0 & -2 & 0 & 1 \end{bmatrix} \sim_{\substack{-5R_2 + R_3 \to R_3 \\ 2R_2 + R_4 \to R_4}} \begin{bmatrix} 5 & 10 & 3 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -9 & -51 \\ 0 & 0 & 0 & 9 \end{bmatrix} \sim_{\frac{1}{9}R_4 \to R_4} \begin{bmatrix} 5 & 10 & 3 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -9 & -51 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim_{\substack{-7R_4 + R_1 \to R_1 \\ -4R_4 + R_2 \to R_2 \\ 51R_4 + R_3 \to R_3}} \begin{bmatrix} 5 & 10 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim_{-\frac{1}{9}R_3 \to R_3} \begin{bmatrix} 5 & 10 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim_{-3R_3 + R_1 \to R_1} \begin{bmatrix} 5 & 10 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim_{-10R_2 + R_1 \to R_1} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim_{\frac{1}{5}R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) No solution since there is a leading 1 in the constant column

$$\begin{aligned} 2y &= 8 \\ -6y &= 3 \\ 3x + 7y &= -2 \\ 5x + 10y + 3z &= 7 \end{aligned}$$

c) Only the trivial solution since the number of leading one is equal the number of variable.

$$\begin{aligned} 2x_2 + 8x_4 &= 0 \\ -6x_2 + 3x_4 &= 0 \\ 3x_1 + 7x_2 - 2x_4 &= 0 \\ 5x_1 + 10x_2 + 3x_3 + 7x_4 &= 0 \end{aligned}$$

d) By the equivalence thm it is possible since the RREF of A is I

f) false

g) true

Alternate answer for 1a.

$$A \sim \cdots \sim \begin{bmatrix} 5 & 10 & 3 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -9 & -51 \\ 0 & 0 & 0 & 9 \end{bmatrix} = B$$

$-5R_2 + R_3 \to R_3$

$2R_2 + R_4 \to R_4$

$Bx = 0$ has only the trivial solution since #var = # leading 1. Hence by the equivalence theorem the RREF of A is I.

Question 2. Consider the matrices:

$$A = [a_{ij}]_{3\times3},\quad B = [b_{ij}]_{2\times3},\quad C = [c_{ij}]_{3\times2},\quad D = [d_{ij}]_{2\times2},\quad E = [e_{ij}]_{3\times6}$$

where $a_{ij} = i - j$, $b_{ij} = (-1)^i 2 + (-1)^j 3$, $c_{ij} = i + j$, $d_{ij} = (ij)^2$, $e_{ij} = i + j$. Evaluate the following if possible, justify.

a. *(2 marks) AB*

$$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -5 & 1 & -5 \\ -1 & 5 & -1 \end{bmatrix}$$

is not defined since the # of columns of A is not equal to the # of rows of B

b. *(2 marks) AE*

$$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} -11 & -14 & -17 & -20 & -23 & -26 \\ -2 & -2 & -2 & -2 & -2 & -2 \\ 7 & 10 & 13 & 16 & 19 & 22 \end{bmatrix}$$

Consider the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix},\ B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix},\ D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix},\ E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Evaluate the following if possible, justify.

a. *(2 marks)* $\text{trace}(C)BA - \text{trace}(D)EE^T$ $= 1 \cdot \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix} - (-4) \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -11 & -8 \\ 4 & -3 \end{bmatrix} + 4 \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 16 & 33 \end{bmatrix}$$

b. *(2 marks)* $C(BA + 2I)$ $= \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \left(\begin{bmatrix} -11 & -8 \\ 4 & -3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$$= \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -9 & -8 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -10 \\ -45 & -40 \end{bmatrix}$$

Question 3. Let A, B and C denote $n \times n$ matrices.

a. *(2 marks)* Prove or disprove: If $AB = AC$ then $B = C$.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

then $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but

$B \neq C$. $\therefore$ statement is false.

b. *(2 marks)* Prove or disprove: If A is invertible and $AB = AC$ then $B = C$.

Premise: • A is invertible $\Rightarrow AA^{-1} = I = A^{-1}A$
• $AB = AC$

Conclusion: $B = C$

$$AB = AC$$
$$A^{-1}A\,B = A^{-1}AC$$
$$I\,B = I\,C$$
$$B = C$$

[redacted]

Question 4. Let A and B denote $n \times n$ invertible matrices.

a. *(2 marks)* Show that $A^{-1} + B^{-1} = A^{-1}(A+B)B^{-1}$

$RHS = A^{-1}(A+B)B^{-1}$
$= (A^{-1}A + A^{-1}B)B^{-1}$
$= (A^{-1}AB^{-1} + A^{-1}BB^{-1}) = IB^{-1} + A^{-1}I = B^{-1} + A^{-1} = A^{-1} + B^{-1} = LHS$

b. *(3 marks)* If $A+B$ is also invertible, show that $A^{-1}+B^{-1}$ is invertible and find a formula for $(A^{-1}+B^{-1})^{-1}$. *(Hint: use part a.)*

Premise: • A & B are invertible $\Rightarrow AA^{-1} = I = A^{-1}A$, $BB^{-1} = I = B^{-1}B$
• $A+B$ is invertible $\Rightarrow (A+B)(A+B)^{-1} = I = (A+B)^{-1}(A+B)$

Conclusion: • $A^{-1}+B^{-1}$ is invertible

Need to find C such that ① $(A^{-1}+B^{-1})C = I$ and ② $C(A^{-1}+B^{-1}) = I$

①: $(A^{-1}+B^{-1})C = A^{-1}(A+B)B^{-1}C$, let $C = (A^{-1}(A+B)B^{-1})^{-1} = (B^{-1})^{-1}(A+B)^{-1}(A^{-1})^{-1}$
$= B(A+B)^{-1}A$

$= A^{-1}(A+B)B^{-1}B(A+B)^{-1}A$
$= A^{-1}(A+B)\,I\,(A+B)^{-1}A = A^{-1}(A+B)(A+B)^{-1}A = A^{-1}IA = A^{-1}A = I$

Similarly for ②. $\therefore$ $A^{-1}+B^{-1}$ is invertible and its inverse is $B(A+B)^{-1}A$.

[redacted]

Alternate answer for 3b.

Since A, B and A+B is invertible it follows that A^{-1}, B^{-1} and $(A+B)^{-1}$ is invertible. And the product

$$A^{-1}(A+B)B^{-1}$$

of invertible matrices is invertible.

By part a.
$A^{-1}+B^{-1}$ is invertible since

$$A^{-1}+B^{-1} = A^{-1}(A+B)B^{-1}$$

and

$$(A^{-1}+B^{-1})^{-1} = (A^{-1}(A+B)B^{-1})^{-1}$$
$$= B(A+B)^{-1}A$$

Question 5. Given the 5 × 5 *Hilbert matrix* and its inverse

$$H=\begin{bmatrix}\frac{1}{1}&\frac{1}{2}&\frac{1}{3}&\frac{1}{4}&\frac{1}{5}\\\frac{1}{2}&\frac{1}{3}&\frac{1}{4}&\frac{1}{5}&\frac{1}{6}\\\frac{1}{3}&\frac{1}{4}&\frac{1}{5}&\frac{1}{6}&\frac{1}{7}\\\frac{1}{4}&\frac{1}{5}&\frac{1}{6}&\frac{1}{7}&\frac{1}{8}\\\frac{1}{5}&\frac{1}{6}&\frac{1}{7}&\frac{1}{8}&\frac{1}{9}\end{bmatrix},\quad H^{-1}=\begin{bmatrix}25&-300&1050&-1400&630\\-300&4800&-18900&26880&-12600\\1050&-18900&79380&-117600&56700\\-1400&26880&-117600&179200&-88200\\630&-12600&56700&-88200&44100\end{bmatrix}\quad\text{and } A=\begin{bmatrix}\frac{1}{5}&\frac{1}{6}&\frac{1}{7}&\frac{1}{8}&\frac{1}{9}\\1&\frac{2}{3}&\frac{1}{2}&\frac{2}{5}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{4}&\frac{1}{5}&\frac{1}{6}&\frac{1}{7}\\\frac{1}{4}&\frac{1}{5}&\frac{1}{6}&\frac{1}{7}&\frac{1}{8}\\\frac{1}{1}&\frac{1}{2}&\frac{1}{3}&\frac{1}{4}&\frac{1}{5}\end{bmatrix}.$$

a. *(2 marks)* Show that H is row equivalent to A by finding two elementary matrices E_i such that $E_2E_1H = A$.

A can be obtained from H by performing $R_1 \leftrightarrow R_5$ and $2R_2 \to R_2$.

$$\therefore E_1: I \sim R_1 \leftrightarrow R_5 \quad \begin{bmatrix}0&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\end{bmatrix} = E_1$$

$$E_2: I \sim 2R_2 \to R_2 \quad \begin{bmatrix}1&0&0&0&0\\0&2&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} = E_2$$

b. *(3 marks)* Find the inverse of A. *(Hint: use part a.)*

$$E_2E_1H = A$$

$$A^{-1} = (E_2E_1H)^{-1}$$

$$= H^{-1}E_1^{-1}E_2^{-1} \qquad \text{note: } E_1 = E_1^{-1}$$

$$= H^{-1}E_1\begin{bmatrix}1&0&0&0&0\\0&\frac{1}{2}&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$= H^{-1}\begin{bmatrix}0&0&0&0&1\\0&1/2&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\1&0&0&0&0\end{bmatrix} = \begin{bmatrix}630&-150&1050&-1400&25\\-12600&2400&-1890&26880&-300\\56700&-9450&79380&-117600&1050\\-88200&13440&-117600&179200&-1400\\44100&-6300&56700&-88200&630\end{bmatrix}$$

Question 6.[1] *(5 marks)* Let

$$A = \begin{bmatrix} 1 & 1 & a \\ 1 & a & a \\ a & a & a \\ a & a & a^2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a^2 - 2a \end{bmatrix}$$

a. For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have no solution?

b. For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have a unique solution?

c. For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions?

$$\left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 1 & a & a & 0 \\ a & a & a & 0 \\ a & a & a^2 & a^2-2a \end{array}\right] \sim \begin{array}{l} -R_1+R_2 \to R_2 \\ -aR_1+R_3 \to R_3 \\ -aR_1+R_4 \to R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 0 & -1 \\ 0 & 0 & a-a^2 & -a \\ 0 & 0 & 0 & a^2-3a \end{array}\right]$$

a) For the system to have no solutions there needs to be a leading entry in the constant column. The possibilities are:

① $a^2-3a \neq 0$ or ② $a - a^2 = 0$ and $-a \neq 0$ or ③ $a-1=0$

① $a(a-3) \neq 0$, $\therefore a \neq 0$ or $a \neq 3$

② $a(1-a)=0$, $a=0$ or $a=1$; $a \neq 0$; $\therefore a = 1$

③ $a=1$

$\therefore$ no solution if $a \neq 0$, $a \neq 3$, $a=1$

b) Unique solution: # leading 1's in var column = # var. = 3. Note there can not be a leading 1 in the var column.
This is the case if $a-1 \neq 0$ and $a-a^2 \neq 0$ and $a^2-3a=0$

$a \neq 1$; $a(1-a) \neq 0$: $a \neq 0$, $a \neq 1$; $a(a-3)=0$: $a=0$, $a=3$

$\therefore$ unique solution if $a=3$

c) Infinitely many solutions: # leading 1 in var. column < # var = 3.
This is the case if $a^2-3a=0$ and $a-a^2=0$ and $-a=0$ and $a-1 \neq 0$

$\therefore$ infinitely many solutions if $a=0$

[1] From a John Abbot Final Examination

Question 7. *(5 marks)* Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

note: $E_3^{-1} = E_3$

$$E_3\left(\text{trace}(E_1)X^T + E_2A\right) = A$$

solve for X, if possible.

$$E_3^{-1}E_3(3X^T + E_2A) = E_3^{-1}A$$

$$3X^T + E_2A = E_3A$$

$$3X^T = E_3A - E_2A$$

$$3X^T = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -6 & -6 & -6 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

$$3X^T = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 8 & 8 & 8 & 8 \\ -2 & -2 & -2 & -2 \end{bmatrix}$$

$$X^T = \begin{bmatrix} 2/3 & 2/3 & 2/3 & 2/3 \\ 8/3 & 8/3 & 8/3 & 8/3 \\ -2/3 & -2/3 & -2/3 & -2/3 \end{bmatrix}$$

$$X = \begin{bmatrix} 2/3 & 8/3 & -2/3 \\ 2/3 & 8/3 & -2/3 \\ 2/3 & 8/3 & -2/3 \\ 2/3 & 8/3 & -2/3 \end{bmatrix}$$

Bonus Question. *(5 marks)*
Prove: If E is an elementary matrix then E^T is an elementary matrix.

Suppose E is obtained by $I \sim R_i \leftrightarrow R_j$ E then the 1 in ii location is in the ji location. Similarly the 1 in the jj location is in ij location. Performing the transpose on E the will interchange the 1 in the ji and ij location. Hence $E^T = E$. ∴ E^T is an elementary matrix.

Suppose E is obtained by $I \sim cR_i \rightarrow R_i$ E then the 1 in the ii location is a c. Performing the transpose on E will not change the matrix since all the entries are on the main diagonal and zeros everywhere else. ∴ $E^T = E$. ∴ E^T is an elementary matrix.

Suppose E is obtained by $I \sim cR_i + R_j \rightarrow R_j$ E then E is the identity plus a c entry in the ji location. Performing the transpose on E, we obtain the identity plus a c entry in the location ji. Hence we can obtain E^T by $I \sim cR_j + R_i \rightarrow R_i$ E^T. ∴ E^T is an elementary matrix.