Dawson College: **Linear Algebra**: 201-NYC-05-S05: Winter 2017

Name: Y. Lamontagne

Test 2

This test is graded out of 46 marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531XG or Sharp EL-531X calculator. Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

Question 1. Given

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -3 & -4 \\ 2 & 3 & 0 & 0 \\ 0 & 4 & 5 & 0 \end{bmatrix}.$$

$$|A| \overset{-2R_1+R_3\to R_3}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -3 & -4 \\ 0 & -1 & -6 & -8 \\ 0 & 4 & 5 & 0 \end{vmatrix} \overset{\substack{-2R_3+R_2\to R_2 \\ 4R_3+R_4\to R_4}}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 9 & 12 \\ 0 & -1 & -6 & -8 \\ 0 & 0 & -19 & -32 \end{vmatrix}$$

a. *(5 marks)* Evaluate $\det(A)$.

b. *(5 marks)* If M is a 4×4 matrix such that $\det(M) = 2$ then evaluate $\det(\det(M)\operatorname{adj}(5M^TA^{-1}))$. Justify!

$$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}$$

$$= 1\,C_{11}$$

$$= \begin{vmatrix} 0 & 9 & 12 \\ -1 & -6 & -8 \\ 0 & -19 & -32 \end{vmatrix} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = -1(-1)\begin{vmatrix} 9 & 12 \\ -19 & -32 \end{vmatrix} = -60$$

b) $\det(\det(M)\operatorname{adj}(5M^TA^{-1}))$

$$= (\det M)^4 \det(\operatorname{adj}(5M^TA^{-1}))$$

$$= 2^4(\det(5M^TA^{-1}))^{4-1}$$

$$= 2^4(5^4\det(M^TA^{-1}))^3$$

$$= 2^4 5^{12}(\det(M^TA^{-1}))^3$$

$$= 2^4 5^{12}(\det M^T \det A^{-1})^3$$

$$= 2^4 5^{12}(\det M)^3\left(\frac{1}{\det A}\right)^3$$

$$= 2^4 5^{12}(2)^3\left(\frac{1}{-60}\right)^3$$

$$= -\frac{2^7 5^{12}}{60^3}$$

$$= -\frac{2^7 5^{12}}{4^3 \cdot 3^3 5^3}$$

$$= -\frac{2\cdot 5^9}{3^3}$$

Question 2. *(3 marks)* Prove: If $AX = 0$ for some $X \neq 0$, then $\det(A) = 0$.

Proof by contradiction: Suppose $\det A \neq 0$ then A is invertible

i.e. $\exists A^{-1}$ s.t. $A^{-1}A = I = AA^{-1}$

Suppose $X \neq 0$ s.t $AX = 0$ (by the premise)

$$AX = 0$$

$$A^{-1}AX = A^{-1}0$$

$$X = 0 \quad \lightning$$

$\therefore \det A = 0$

Question 3. *(3 marks)* Prove or disprove: There does not exist an $n \times n$ matrix A where n is odd such that $A^2 + I = 0$.

Proof by contradiction: Suppose $\exists A$ s.t. $A^2 + I = 0$

$$A^2 + I = 0$$

$$A^2 = -I$$

$$\det(A^2) = \det(-I)$$

$$(\det A)^2 = (-1)^n \det I$$

$$(\det A)^2 = (-1)^n \cdot 1$$

$$(\det A)^2 = -1 \quad \text{since } n \text{ is odd.}$$

$\lightning$

$\therefore$ No A exists s.t. $A^2 + I = 0$

Question 4. *(3 marks)* Given $A = [a_{ij}]_{n\times n}$ and the cofactors of A, C_{ij}. Show that $a_{11}C_{21} + a_{12}C_{22} + \ldots + a_{1n}C_{2n} = 0$.

Lets determine B s.t. $|B| = a_{11}C_{21} + a_{12}C_{22} + \ldots + a_{1n}C_{2n}$

suppose a cofactor expansion along the first row of B

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

$$|B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = 0 \quad \text{Since } R_1 = R_2.$$

first column of C_{22}

from C_{21}

Question 5. Given the line $(x,\ y,\ z) = (1+3t,\ 2+2t,\ 3+t)$ where $t \in \mathbb{R}$ and point $P(3,\ 2,\ 1)$.

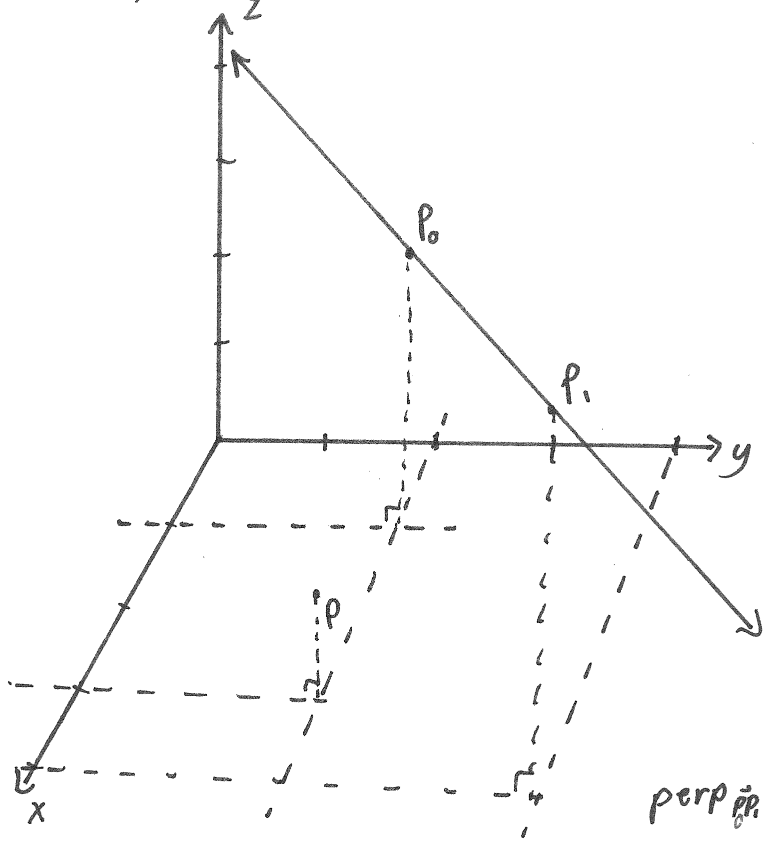
a. *(2 marks)* Sketch the given line using P_0 the point on the line when $t = 0$ and P_1 the point on the line when $t = 1$, also sketch P.

b. *(5 marks)* Using projections find area of the triangle defined by the points P_0, P_1 and P.

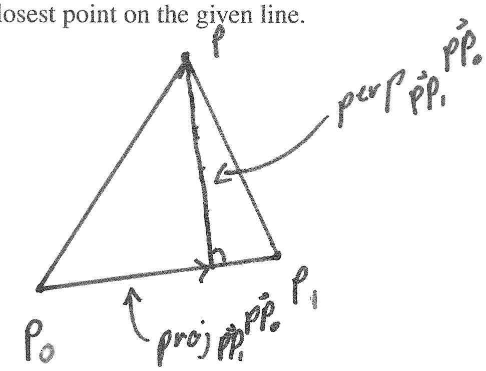
c. *(2 marks)* Find the equation of the line which passes through the point P and the closest point on the given line.

a) $t=0,\ P_0(1,2,3)$

$t=1,\ P_1(4,4,4)$



b)



$$\text{Area} = \frac{\text{base}\cdot\text{height}}{2} = \frac{\|\vec{P_0P_1}\|\,\|\operatorname{perp}_{\vec{P_0P_1}}\vec{P_0P}\|}{2}$$

$$\vec{P_0P_1} = P_1 - P_0 = (4,4,4)-(1,2,3) = (3,2,1)$$

$$\vec{P_0P} = P - P_0 = (3,2,1)-(1,2,3) = (2,0,-2)$$

$$\operatorname{perp}_{\vec{P_0P_1}}\vec{P_0P} = \vec{P_0P} - \operatorname{proj}_{\vec{P_0P_1}}\vec{P_0P}$$

$$= (2,0,-2) - \frac{\vec{P_0P}\cdot\vec{P_0P_1}}{\vec{P_0P_1}\cdot\vec{P_0P_1}}\vec{P_0P_1}$$

$$= (2,0,-2) - \frac{4}{14}(3,2,1)$$

$$= (2,0,-2) - \frac{2}{7}(3,2,1)$$

$$= \left(\frac{8}{7}, \frac{-4}{7}, \frac{-16}{7}\right) = \frac{4}{7}(2,-1,-4)$$

$$A = \frac{\|(3,2,1)\|\,\|\frac{4}{7}(2,-1,-4)\|}{2}$$

$$= \frac{\sqrt{14}\ \frac{4}{7}\sqrt{4+1+16}}{2} = \frac{4\sqrt{2}\sqrt{7}\sqrt{3}\sqrt{7}}{14}$$

$$= \frac{4\sqrt{6}\cdot 7}{14}$$

$$= 2\sqrt{6}$$

c) (x,y,z)

$= (3,2,1) + t\operatorname{perp}_{\vec{P_0P_1}}\vec{P_0P} \qquad t \in \mathbb{R}.$

$= (3,2,1) + t(2,-1,-4)$

Question 6. *(4 marks)* Express $B = \begin{bmatrix} c & d \\ ra+2c & rb+2d \end{bmatrix}$ as $E_3E_2E_1A$ where E_i are elementary matrices and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then find the determinant of B by using the expression $E_3E_2E_1A$.

$$B \sim R_1 \leftrightarrow R_2 \begin{bmatrix} ra+2c & rb+2d \\ c & d \end{bmatrix}$$

$$\sim -2R_2+R_1 \to R_1 \begin{bmatrix} ra & rb \\ c & d \end{bmatrix}$$

$$\sim \tfrac{1}{r}R_1 \to R_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$$\det(B) = \det(E_3E_2E_1A)$$
$$\det(B) = \det(E_3)\det(E_2)\det(E_1)\det A$$
$$= (-1)(1)r(2)$$
$$= -2r.$$

So $\quad F_3F_2F_1B = A$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim R_1 \leftrightarrow R_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = F_1$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim -2R_2+R_1 \to R_1 \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = F_2$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \tfrac{1}{r}R_1 \to R_1 \begin{bmatrix} 1/r & 0 \\ 0 & 1 \end{bmatrix} = F_3$$

$$B = F_1^{-1}F_2^{-1}F_3^{-1}A$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix}A$$

$$B = E_3E_2E_1A$$

Question 7. *(3 marks)* Prove: If $\vec{u}$ and $\vec{v}$ are orthogonal then $||\vec{u}+\vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2$

$$||\vec{u}+\vec{v}||^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$
$$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$
$$= ||\vec{u}||^2 + 0 + 0 + ||\vec{v}||^2$$ since $\vec{u}\cdot\vec{v} = 0$ since $\vec{u}$ and $\vec{v}$ are orthogonal.
$$= ||\vec{u}||^2 + ||\vec{v}||^2$$

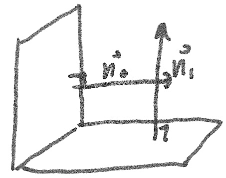
Question 8. Given two planes $x+y+z=1$ and $x+2y+3z=4$.

a. *(2 marks)* Determine whether the two planes are perpendicular, parallel or neither. Justify!

b. *(3 marks)* Find the intersection between the two planes if it exists.

c. *(3 marks)* Find the smallest angle between the two planes.

a) Two planes are $\perp$ iff $\vec{n}_1 \cdot \vec{n}_2 = 0$

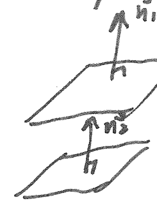


$\vec{n}_1 \cdot \vec{n}_2 = (1,1,1)\cdot(1,2,3)$
$= 6 \neq 0$

$\therefore$ the two planes are not $\perp$

$\therefore$ neither

Two planes are $\parallel$ iff $\vec{n}_1 = k\vec{n}_2$



$\vec{n}_1 = (1,1,1)$ is not a multiple of $\vec{n}_2 = (1,2,3)$

$\therefore$ the two planes are not parallel!

b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

$\sim_{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim_{-R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

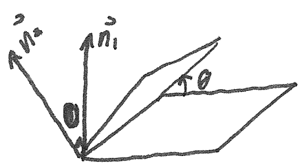
Let $z = t$

$x = -2+t$

$y = 3-2t$

$\therefore (x,y,z) = (-2,3,0) + t(1,-2,1)$.

c)



$\vec{n}_1 \cdot \vec{n}_2 = \|\vec{n}_1\|\|\vec{n}_2\| \cos\theta$

$1+2+3 = \sqrt{3}\sqrt{14} \cos\theta$

$\dfrac{6}{\sqrt{3}\sqrt{14}} = \cos\theta$

$\theta = 22^{\circ}$

Bonus Question. *(5 marks)*
Prove: If A and B are two matrices such that $A+B=AB$ then A and B commute.

Given $A+B=AB$ we get $A = AB - B = (A-I)B$ and $AB - A - B = 0$

$B = AB - A = A(B-I)$

So $BA = A[(B-I)(A-I)]B$

$= AIB$

$= AB$

note that: $(A-I)(B-I)$

$= \underbrace{AB - A - B}_{} + I$

$= 0 + I$

$= I$

Hence $(A-I)$ and $(B-I)$ are inverses of each other

Solution by Bogdan.