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Test 3

This test is graded out of 46 marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531XG or Sharp EL-531X calculator. Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

Question 1. Given $\vec{u} = \overbrace{(-2,-1,0)}^{\vec{d}_1}$ and $\vec{v} = \overbrace{(1,0,1)}^{\vec{d}_2}$

$$\begin{array}{lcl} \mathcal{L}_1 & : & (x,y,z) = \overbrace{(1,0,1)}^{P_1} \quad + \quad t_1\vec{u} \\ \mathcal{L}_2 & : & (x,y,z) = \underbrace{(-2,-1,2)}_{P_2} \quad + \quad t_2\vec{v} \text{ where } t_1,\ t_2 \in \mathbb{R}. \end{array}$$

a. *(3 marks)* Determine whether the two lines intersect, are parallel or are skew lines.

b. *(3 marks)* Find the shortest distance between the two lines.

c. *(2 marks)* Find the area of the triangle defined by $\vec{u}$ and $\vec{v}$.

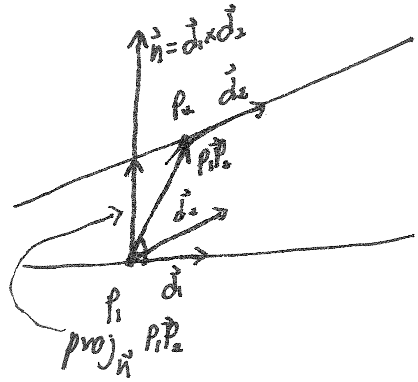
d. *(2 marks)* Find the equation of a line which is orthogonal to the direction of both given lines and passes through $\mathcal{L}_1$. Is the line unique?

a) $\mathcal{L}_1$ and $\mathcal{L}_2$ are not parallel since $\vec{d}_1 \| \vec{d}_2$. Do $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect?

$$\begin{array}{ll} ① & 1-2t_1 = -2+t_2 \\ ② & -t_1 = -1 \\ ③ & 1 = 2+t_2 \end{array} \Rightarrow t_1 = 1 \text{ sub into ①} \quad \begin{array}{l} 1-2(1) = -2+t_2 \\ 1 = t_2 \end{array} \quad \text{check consistency} \quad \begin{array}{l} 1 \overset{?}{=} 2+1 \\ 1 \overset{?}{=} 3 \end{array}$$

Not consistent $\therefore$ no intersection $\therefore$ $\mathcal{L}_1$ and $\mathcal{L}_2$ are skew lines

b)



$\vec{P_1P_2} = P_2 - P_1 = (-2,-1,2) - (1,0,1) = (-3,-1,1)$

$\vec{n} = \vec{d}_1 \times \vec{d}_2 = \left(\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} -2 & 1 \\ -1 & 0 \end{vmatrix} \right) = (-1, 2, 1)$

$$\begin{array}{cc} -2 & 1 \\ -1 & 0 \\ 0 & 1 \end{array}$$

$$\begin{aligned} \text{distance} &= \|\text{proj}_{\vec{n}} \vec{P_1P_2}\| = \left\| \frac{\vec{P_1P_2} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \right\| \\ &= \left\| \frac{-3(-1) + (-1)(2) + (1)(1)}{-1(-1) + 2 \cdot 2 + (1)(1)} (1,2,-1) \right\| \\ &= \left\| \frac{2}{6} (1,2,-1) \right\| \\ &= \frac{1}{3} \|(1,2,-1)\| = \frac{1}{3}\sqrt{1^2+2^2+(-1)^2} = \sqrt{\frac{6}{9}} \\ &= \sqrt{\frac{2}{3}} \end{aligned}$$

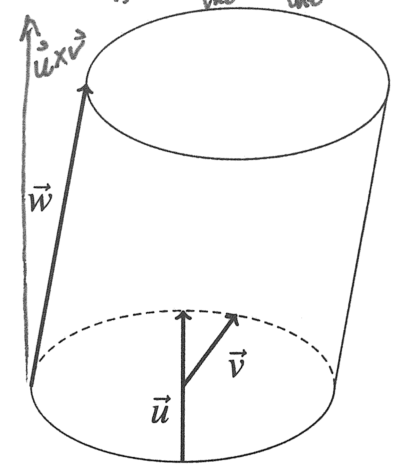
c) $\text{Area} = \dfrac{\|\vec{u} \times \vec{v}\|}{2} = \dfrac{\|(-1,2,1)\|}{2} = \dfrac{\sqrt{6}}{2} = \sqrt{\dfrac{6}{4}} = \sqrt{\dfrac{3}{2}}$

d) $\mathcal{L} : (x,y,z) = (1,0,1) + t\vec{n}$

$= (1,0,1) + t(-1,2,1) \quad t \in \mathbb{R}$

The line is not unique as it can pass through any point of $\mathcal{L}_1$.

Question 2. *(5 marks)* A cylinder where the sides are not perpendicular to the base(the circle) is called an *oblique cylinder*. Given the oblique cyclinder defined by the given vectors $\vec{u} = (2,2,4)$, $\vec{v} = (1,2,1)$ and $\vec{w} = (4,1,3)$. Note from the diagram below that $\vec{w}$ is not perpendicular to the base, that $\vec{v}$ is positioned such that its tail is at the center of the circle and its tip lies on the circle, that $\vec{u}$ is positioned such that the vector passes through the center of the circle while its tail and tip lie on the circle. Find the volume of the oblique cylinder. *(Hint: volume of an oblique cyclinder equal to area of base times height.)*



area of base $= \pi r^2 = \pi (\|\vec{v}\|)^2 = \pi\left(\sqrt{(1)^2+(2)^2+(1)^2}\right)^2$
$= 6\pi$

height $= \|\text{proj}_{\vec{v}\times\vec{u}}\, \vec{w}\|$

$\vec{v}\times\vec{u} = \left(\begin{vmatrix}2&1\\2&4\end{vmatrix}, -\begin{vmatrix}1&1\\2&4\end{vmatrix}, \begin{vmatrix}1&2\\2&2\end{vmatrix}\right)$

$\begin{matrix}1&2\\2&2\\1&4\end{matrix} = (6,-2,-2)$

$= \left\| \frac{(4,1,3)\cdot(6,-2,-2)}{(6,-2,-2)\cdot(6,-2,-2)} (6,-2,-2)\right\|$

$= \left\| \frac{16}{44}(6,-2,-2)\right\|$

$= \left\| \frac{4}{11}(6,-2,-2)\right\| = \frac{4}{11}\|(6,-2,-2)\|$

$= \frac{4}{11}\sqrt{36+4+4}$

$= \frac{4\sqrt{44}}{11}$

$= \frac{8\sqrt{11}}{11}$

Volume = base × height
$= 6\pi \cdot \frac{8\sqrt{11}}{11} = \frac{48\pi\sqrt{11}}{11}$

Question 3. Given two planes:

$$\begin{array}{lcrcrcrcr} \mathscr{P}_1 & : & x & - & 2y & + & z & = & 1 \\ \mathscr{P}_2 & : & -4x & + & 8y & - & 4z & = & -4 \end{array}$$

a. *(1 mark)* Give a geometrical argument to explain why the intersection of the two planes is a plane.

b. *(2 marks)* Find the solution set of the associated homogeneous linear system.

c. *(2 marks)* Show that the solution set of the associated homegeneous linear system is orthogonal to the rows of the coefficient matrix of the system.

d. *(2 marks)* Give a geometrical interpretation to part c).

a) Both planes have the same inclination since their normals are parallel and have (1,0,0) in common. ∴ both planes are identical and the intersection is the plane.

b) $\begin{bmatrix}1&-2&1&0\\-4&8&-4&0\end{bmatrix} \sim_{+4R_1+R_2\to R_2} \begin{bmatrix}1&-2&1&0\\0&0&0&0\end{bmatrix}$ Let $y=s$, $z=t$, $x=2s-t$, $s,t\in\mathbb{R}$

∴ $\overbrace{(x,y,z)}^{\vec{x}} = (2s-t, s, t)$
$= s(2,1,0)+t(-1,0,1)$

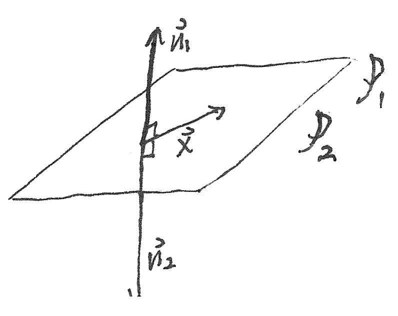
c) Let $\vec{r}_1 = (1,-2,1)$ and $\vec{r}_2 = -4\vec{r}_1$ be the rows of the coefficient matrix

$\vec{r}_1\cdot\vec{x} = (1,-2,1)\cdot(2s-t, s, t)$
$= 1\cdot(2s-t) - 2(s) + t$
$= 0$

$\vec{r}_2\cdot\vec{x} = (-4\vec{r}_1)\cdot\vec{x} = -4(\vec{r}_1\cdot\vec{x}) = -4(0) = 0$

∴ $\vec{r}_i \perp \vec{x}$

d) The rows of the coefficient matrix are the normals of the planes Hence the vector that lie on the plane are ⊥ to the normals



Question 4. A function f is said to be *odd* if $f(-x) = -f(x)$. Let $V = \{f \mid f : \mathbb{R} \to \mathbb{R}$ and f is an odd function$\}$ with the following vector addition and scalar multiplication:

$$(f+g)(x) = f(x)g(x) \text{ and } (kf)(x) = (f(x))^k$$

a. *(3 marks)* Is the zero vector an element of V? Justify.

b. *(3 marks)* Determine whether the following axiom holds: $(rs)f = r(sf)$ where $r,\ s \in \mathbb{R}$ and $f \in V$

a) Let $f \in V$ and $0 \in \{f : | f : \mathbb{R} \to \mathbb{R}\}$ $(f+0)(x) = f(x)$

$f(x)0(x) = f(x) \Rightarrow 0(x) = 1$

$\to 0(x) \notin V$ since $0(-x) = 1 = 0(x)$.

b) Let $f \in V$ and $r, s \in \mathbb{R}$

LHS $= ((rs)f)(x)$
$= (f(x))^{rs}$

RHS $= (r(sf))(x) = ((sf)(x))^r$
$= ((f(x))^s)^r = (f(x))^{rs} = $ LHS

$\therefore$ the axiom holds

Question 5. *(3 marks)* Determine whether $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid (ab)^2 = (cd)^2 \right\}$ is a subspace of $\mathscr{M}_{2\times 2}$.

Let $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W$ since $(1 \cdot 0)^2 \stackrel{?}{=} (0 \cdot 1)^2$
$0 = 0$

$M_2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \in W$ since $(0 \cdot 1)^2 \stackrel{?}{=} (2 \cdot 0)^2$
$0 = 0$

But $M_1 + M_2 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \notin W$ since $(1 \cdot 1)^2 \neq (2 \cdot 1)^2$

Question 6. Given $\vec{u} = (1,2,3), \vec{v} = (-3,-6,\lambda) \in \mathbb{R}^3$ and the set $S = \{\vec{u}, \vec{v}, \vec{u} \times \vec{v}\}$.

a. *(2 marks)* Find the value(s), if any, of λ for which S spans $\mathbb{R}^3$

b. *(2 marks)* Find the value(s), if any, of λ for which S spans a plane.

c. *(2 marks)* Find the value(s), if any, of λ for which S spans a line.

a) Notice that if $\lambda = -9$, $\vec{u} \parallel \vec{v}$ which implies that $\vec{u} \times \vec{v} = \vec{0}$. Since $\dim(\mathbb{R}^3) = 3$ for S to span $\mathbb{R}^3$, S needs to have 3 linearly independent vectors. Not the case if $\lambda = -9$. If $\lambda \neq -9$ then $\vec{u}$ and $\vec{v}$ are not parallel and $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. Hence linearly independent since it is not colinear, coplanar. $\therefore$ spans $\mathbb{R}^3$

b) if $\lambda = -9$ then by a) $\operatorname{span}(S) = \operatorname{span}(\{(1,2,3)\})$ since $\vec{w} = \vec{0}$ and $\vec{v} = -3\vec{u}$

$= \{t(1,2,3) \mid t \in \mathbb{R}\}$

Hence spans a line

$\therefore$ no values of λ such that $\operatorname{span}(S)$ is a plane.

Question 7. Given $W = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + 2a_1 + 3a_2 + 4a_3 = 0 \text{ and } a_1 + a_2 + a_3 = 0\}$ a subspace of P_3.

a. *(3 marks)* Find a basis B for W.

b. *(1 mark)* State the $\dim(W)$ and $\dim(P_3)$.

a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \sim_{-2R_1 + R_2 \to R_1} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$ Let $a_2 = s$ $\quad s,t \in \mathbb{R}$

$a_3 = t$

$a_0 = -s - 2t$

$a_1 = -s - t$

$\therefore p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$= (-s-2t) + (-s-t)x + sx^2 + tx^3 = s(-1 - x + x^2) + t(-2 - x + x^3)$

$\therefore$ W is spanned by $\{-1 - x + x^2, -2 - x + x^3\}$ and is linearly independent since the two polynomial are not multiple of each other. $\therefore$ its a basis of W

b) $\dim(W) = 2$ since the basis has 2 vectors

$\dim(P_3) = 4$ since a basis for P_3 is $\{1, x, x^2, x^3\}$

Bonus Question. *from Wikipedia (3 marks)* Discuss the following: The barber is the "one who shaves all those, and those only, who do not shave themselves." The question is, does the barber shave himself?