**Dawson College**: **Linear Algebra**: 201-NYC-05-S02: Winter 2018

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# Test 1

This test is graded out of 48 marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531 calculator. Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

**Question 1.** Given

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ -3 & 2 & 1 \end{bmatrix}.$$

a. *(5 marks)* Find all $3 \times 3$ lower triangular matrix $B$ such that $AB = 0$.

b. *(1 marks)* List two such matrix $B$ from part a.

b) $s=t=0$ $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $s=t=2$ $B = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$

Let $B = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 2a+2b+d & 2c+e & f \\ -3a+2b+d & 2c+e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $f=0$ and we have the following system

$$\begin{aligned} 2a + 2b + d &= 0 \\ -3a + 2b + d &= 0 \\ 2c + e &= 0 \\ 2c + e &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|cc} 2 & 2 & 0 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right] \sim \begin{array}{l} 2R_2 \to R_2 \\ \\ -R_3+R_4 \to R_4 \end{array} \left[\begin{array}{cccccc} 2 & 2 & 0 & 1 & 0 & 0 \\ -6 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{l} 3R_1+R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array} \left[\begin{array}{cccccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 10 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{l} -\frac{1}{5}R_2+R_1 \to R_1 \\ \frac{1}{10}R_2 \to R_2 \end{array} \left[\begin{array}{cccccc} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \frac{1}{2}R_1 \to R_1 \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $d=s$, $e=t$ $s,t \in \mathbb{R}$

then $a=0$, $b=-\frac{1}{2}s$, $c=-\frac{1}{2}t$, $d=s$, $e=t$, $f=0$

$\therefore$ $B = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2}s & -\frac{1}{2}t & 0 \\ s & t & 0 \end{bmatrix}$

**Question 2.** Given the matrix $A = \begin{bmatrix} 1 & x \\ 0 & -x \end{bmatrix}$,

a. *(2 marks)* Calculate $A^2$ and $A^3$

$$A^2 = AA = \begin{bmatrix} 1 & x \\ 0 & -x \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & -x \end{bmatrix} = \begin{bmatrix} 1 & x - x^2 \\ 0 & x^2 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 1 & x \\ 0 & -x \end{bmatrix}\begin{bmatrix} 1 & x - x^2 \\ 0 & x^2 \end{bmatrix} = \begin{bmatrix} 1 & x - x^2 + x^3 \\ 0 & -x^3 \end{bmatrix}$$

$$A^4 = AA^3 = \begin{bmatrix} 1 & x \\ 0 & -x \end{bmatrix}\begin{bmatrix} 1 & x - x^2 + x^3 \\ 0 & -x^3 \end{bmatrix} = \begin{bmatrix} 1 & x - x^2 + x^3 - x^4 \\ 0 & x^4 \end{bmatrix}$$

b. *(2 marks)* Give an expression for $A^n$, where $n$ is any natural number.

$$A^n = \begin{bmatrix} 1 & x - x^2 + x^3 - x^4 + \dots + (-1)^{n+1} x^n \\ 0 & (-1)^n x^n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sum_{i=1}^{n} (-1)^{n+1} x^n \\ 0 & (-1)^n x^n \end{bmatrix}$$

c. *(3 **bonus** marks)* Given the matrix $Q = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, and $|x| < 1$ calculate $\lim_{n\to\infty}(Q + A^n)$ $= \lim_{n\to\infty} \left( \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & \sum_{i=1}^{n} (-1)^{n+1} x^n \\ 0 & (-1)^n x^n \end{bmatrix} \right.$

$$= \lim_{n\to\infty} \begin{bmatrix} 1 & 1 + \sum_{i=1}^{n} (-1)^{n+1} x^n \\ 0 & 1 + (-1)^n x^n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 + \frac{x}{1+x} \\ 0 & 1 \end{bmatrix} \text{ if } |x| < 1$$

see

$$\lim_{n\to\infty}\left(1 + \sum_{i=1}^{n} (-1)^{n+1} x^n\right)$$

$$= \lim_{n\to\infty}\left(1 + x \sum_{i=1}^{n} (-1)^{n-1} x^{n-1}\right)$$

$$= \lim_{n\to\infty}\left(1 + x \sum_{i=1}^{n} (-x)^{n-1}\right)$$

$$= 1 + x \frac{1}{1+x} = 1 + \frac{x}{1+x}$$

since $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^n$ if $|x| < 1$

**Question 3.** *(5 marks)* Express $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$ and its inverse as products of elementary matrices.

$A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \sim_{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \sim_{-3R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$I \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_1$, $I \sim_{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = E_2$, $I \sim_{-3R_2 + R_1 \to R_1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = E_3$

$\because E_3 E_2 E_1 A = I \quad \therefore A^{-1} = E_3 E_2 E_1$

$E_1^{-1} = E_1$, $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and we have $E_3 E_2 E_1 A = I$

$$(E_3 E_2 E_1)^{-1} E_3 E_2 E_1 A = (E_3 E_2 E_1)^{-1} I$$
$$IA = E_1^{-1} E_2^{-1} E_3^{-1}$$
$$\therefore A = E_1^{-1} E_2^{-1} E_3^{-1}$$

**Question 4.** *(4 marks)* Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of $n$ linear equations in $n$ unknowns, and let $Q$ be an invertible $n \times n$ matrix. Prove that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

[$\Rightarrow$] premise: $Ax=0$ has only the trivial solution, $Q$ is invertible
conclusion: $(QA)x=0$ has only the trivial solution

By the premise and the equivalence theorem we can conclude that $A$ is invertible. Then $QA$ is invertible since the product of two invertible matrix is invertible. Again by the equivalence theorem we have that $(QA)x=0$ has only the trivial solution since $QA$ is invertible

[$\Leftarrow$] premise: $(QA)x=0$ has only the trivial solution, $Q$ is invertible
conclusion: $Ax=0$ has only the trivial solution

By the premise and equivalence theorem we can conclude that $QA$ is invertible. So $(QA)^{-1}QA = I$ which implies that $(QA)^{-1}Q$ is the inverse of $A$. Hence $A$ is invertible and by the equivalence theorem $Ax=0$ has only the trivial solution.

**Question 5.** *(5 marks)* Given

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

solve for $X$, if possible.

$$E_3(E_1X + E_1A) = E_2A$$

$$E_3E_1(X+A) = E_2A$$

$$(E_3E_1)^{-1}E_3E_1(X+A) = (E_3E_1)^{-1}E_2A$$

$$X+A = E_1^{-1}E_3^{-1}E_2A$$

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} - A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -6 & -6 & -6 \\ 3 & 3 & 3 & 3 \end{bmatrix} - A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 & 3 & 3 \\ -6 & -6 & -6 & -6 \\ 1 & 1 & 1 & 1 \end{bmatrix} - A$$

$$= \begin{bmatrix} 3 & 3 & 3 & 3 \\ -6 & -6 & -6 & -6 \\ -5 & -5 & -5 & -5 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ -8 & -8 & -8 & -8 \\ -8 & -8 & -8 & -8 \end{bmatrix}$$

**Question 6.** *(3 marks)* If $A$ is a symmetric $n \times n$ matrix and $P$ is any $m \times n$ matrix, show that $PAP^T$ is symmetric.

Need to show that $(PAP^T)^T = PAP^T$

$$LHS = (PAP^T)^T$$
$$= (P^T)^T A^T P^T$$
$$= P A^T P^T$$
$$= PAP^T \quad \text{since } A^T = A$$

**Question 7.** *(1 mark each)* Circle the correct answer. Given $A$ an $n \times n$ matrix and $k$ a non-zero scalar.

a. $A$ is an elementary matrix obtained by interchanging two rows.

i. $\det(A) = -1$
ii. $\det(A) = 0$
iii. $\det(A) = 1$
iv. $\det(A) = k$
v. $\det(A) = n$
vi. $\det(A) = kn$
vii. $\det(A) = k^n$
viii. $\det(A) = n^k$
ix. The determinant of $A$ is not necessarily equal to one of the other options.

b. $A$ is the reduced row echelon form of an invertible matrix.

i. $\det(A) = -1$
ii. $\det(A) = 0$
iii. $\det(A) = 1$
iv. $\det(A) = k$
v. $\det(A) = n$
vi. $\det(A) = kn$
vii. $\det(A) = k^n$
viii. $\det(A) = n^k$
ix. The determinant of $A$ is not necessarily equal to one of the other options.

c. $A$ is a singular matrix.

i. $\det(A) = -1$
ii. $\det(A) = 0$
iii. $\det(A) = 1$
iv. $\det(A) = k$
v. $\det(A) = n$
vi. $\det(A) = kn$
vii. $\det(A) = k^n$
viii. $\det(A) = n^k$
ix. The determinant of $A$ is not necessarily equal to one of the other options.

d. $A$ is an elementary matrix obtained by adding a $k$ times one row to an other.

i. $\det(A) = -1$
ii. $\det(A) = 0$
iii. $\det(A) = 1$
iv. $\det(A) = k$
v. $\det(A) = n$
vi. $\det(A) = kn$
vii. $\det(A) = k^n$
viii. $\det(A) = n^k$
ix. The determinant of $A$ is not necessarily equal to one of the other options.

e. $A$ is an elementary matrix obtained by multiplying one row by $k$.

i. $\det(A) = -1$
ii. $\det(A) = 0$
iii. $\det(A) = 1$
iv. $\det(A) = k$
v. $\det(A) = n$
vi. $\det(A) = kn$
vii. $\det(A) = k^n$
viii. $\det(A) = n^k$
ix. The determinant of $A$ is not necessarily equal to one of the other options.

f. $A$ is the identity matrix multiplied by $k$.

i. $\det(A) = -1$
ii. $\det(A) = 0$
iii. $\det(A) = 1$
iv. $\det(A) = k$
v. $\det(A) = n$
vi. $\det(A) = kn$
vii. $\det(A) = k^n$
viii. $\det(A) = n^k$
ix. The determinant of $A$ is not necessarily equal to one of the other options.

**Question 8.** *(5 marks)* Compute the determinant below by only using co-factor expansions.

$$\begin{vmatrix} 1 & 2 & 0 & 3 \\ -3 & 2 & 1 & 0 \\ 5 & -9 & 0 & 2 \\ 8 & -7 & -3 & 1 \end{vmatrix} = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43}$$

$$= 0\,C_{13} + 1\cdot(-1)^{2+3}\begin{vmatrix} 1 & 2 & 3 \\ 5 & -9 & 2 \\ 8 & -7 & 1 \end{vmatrix} + 0\,C_{33} + (-3)(-1)^{4+3}\begin{vmatrix} 1 & 2 & 3 \\ -3 & 2 & 0 \\ 5 & -9 & 2 \end{vmatrix}$$

$$= -1\left[1\cdot\begin{vmatrix} -9 & 2 \\ -7 & 1 \end{vmatrix} - 2\begin{vmatrix} 5 & 2 \\ 8 & 1 \end{vmatrix} + 3\begin{vmatrix} 5 & -9 \\ 8 & -7 \end{vmatrix}\right]$$

$$+3\left[3\begin{vmatrix} -3 & 2 \\ 5 & -9 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 5 & -9 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix}\right]$$

$$= -\left[5 - 2(-11) + 3(37)\right] + 3\left[3(17) + 2(8)\right]$$

$$= -138 + 201$$

$$= 63$$

**Question 9.** *(5 marks)* Let $A$ and $B$ be $5\times 5$ matrices such that $(A^3B)\mathrm{adj}(2A) = \frac{1}{4}A^T$ and $\det(B) = 7$. Find $\det(A)$.

$$\det\left((A^3B)\,\mathrm{adj}(2A)\right) = \det\left(\tfrac{1}{4}A^T\right)$$

$$\det(A^3)\det(B)\det(\mathrm{adj}(2A)) = \left(\tfrac{1}{4}\right)^n \det(A^T)$$

$$(\det(A))^3\det(B)\,(\det(2A))^{n-1} = \left(\tfrac{1}{4}\right)^n \det(A)$$

$$(\det(A))^{\cancel{3}\,2}\det(B)\,(2^n\det(A))^{n-1} = \left(\tfrac{1}{4}\right)^n \cancel{\det(A)}$$

$$(\det(A))^2\,(7)\,(2^n)^{n-1}(\det(A))^{n-1} = \frac{1}{4^n}$$

$$(\det(A))^{n+1} = \frac{1}{4^n\cdot 7\cdot (2^n)^{n-1}}$$

$$\det(A) = \sqrt[n+1]{\frac{1}{4^n\cdot 7\cdot (2^n)^{n-1}}}$$

$$\det(A) = \pm\sqrt[6]{\frac{1}{4^5\cdot 7\cdot (2^5)^4}}$$

**Question 10.** *(5 marks)* Prove the following identity by only using properties of determinants and elementary operations *(do not evaluate the determinants directly)*.

$$(k^2-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} g(1-k) & h+i & i \\ d(1-k^2) & (e+f)(1+k) & f(1+k) \\ a(1-k) & b+c & c \end{vmatrix}$$

assuming $K \neq 1, -1$. If $K=-1$ or $1$ then it is clear that $LHS = 0 = RHS$. Since the determinant is multiplied by zero on the LHS and there is a row or column of zeros on the RHS.

$$RHS = \begin{vmatrix} g(1-K) & h+i & i \\ d(1-K)(1+K) & (e+f)(1+K) & f(1+K) \\ a(1-K) & b+c & c \end{vmatrix}$$

$\frac{1}{1-K} C_1 \to C_1$

$$= (1-K)\begin{vmatrix} g & h+i & i \\ d(1+K) & (e+f)(1+K) & f(1+K) \\ a & b+c & c \end{vmatrix}$$

$$= (1-K)(1+K)\begin{vmatrix} g & h+i & i \\ d & e+f & f \\ a & b+c & c \end{vmatrix}$$

$\frac{1}{1+K} R_2 \to R_2$

$-C_3 + C_2 \to C_2$

$$= (1-K)(1+K)\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$$

$$= (-1)(1-K)(1+K)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (K^2-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = LHS$$

$R_1 \leftrightarrow R_3$