Dawson College: **Linear Algebra**: 201-NYC-05-S02: Winter 2018

Test 2

This test is graded out of 50 marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531 calculator. Give the work in full unless otherwise stated, reduce each answer to its simplest exact form. Write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

Question 1. *(2 marks)* After years of studying mathematics ____________________ *(<- write your name)* defined a vector product in $\mathbb{R}^3$ which has the same magnitude as the cross product but the direction of the product is given by the left-hand rule *(defined the same way as the right-hand rule but using the left-hand)*. The name of the product is the *happy product* denoted by ☺ and defined as

$\vec{u}$ ☺ $\vec{v} = (u_1, u_2, u_3)$ ☺ $(v_1, v_2, v_3) =$ *(<- write the correct formula for the happy product)*

Draw a sketch illustrating the left-hand rule and the happy product of two vectors.

Question 2.[1] *(5 marks)* Gandalf the Grey started in the Forest of Mirkwood at a point with coordinates $(3, 3)$ and arrived in the Iron Hills at the point with coordinates $(5, 8)$. If he began walking in the direction of the vector $\vec{v} = 4\vec{i} + 2\vec{j}$ and changes direction only once, when he turns at a right angle, what are the coordinates of the point where he makes the turn.

[1] from WeBWorK

Question 3.[2] *(5 marks)* Given the lines:

$$\begin{array}{lcll} \mathscr{L}_1 & : & (x,y,z) = (1,2,-2) & + \quad t_1(1,2,1) \\ \mathscr{L}_2 & : & (x,y,z) = (2,1,3) & + \quad t_2(1,2,3) \\ \mathscr{L}_3 & : & (x,y,z) = (1,1,1) & + \quad t_3(2,7,3) \text{ where } t_1,\ t_2,\ t_3 \in \mathbb{R}. \end{array}$$

which are all skew line to each other. Find the equation of the line which is parallel to $\mathscr{L}_3$ and which intersects both $\mathscr{L}_1$ and $\mathscr{L}_2$.

[2]from Winter 2017 Final Examination

Question 4. *(5 marks)* Determine whether the two lines intersect, are parallel or are skew lines. Find the shortest distance between the lines using projections.

$$\mathscr{L}_1 : \begin{cases} x = 3t \\ y = -1 + 2t \\ z = 1 + 2t \end{cases} \text{ and } \mathscr{L}_2 : \begin{cases} x = 1 - 6t \\ y = -4t \\ z = 2 - 4t \end{cases}$$

Question 5. [3] Let $\vec{u} = (1,\ 3,\ 1)$ and $\vec{v} = (2,\ 1,\ 1)$.

a. *(3 marks)* Find an equation of the form $ax_1 + bx_2 + cx_3 = d$ for the plane spanned by $\vec{u}$ and $\vec{v}$.

b. *(2 marks)* Show that the line $(x_1,\ x_2,\ x_3) = (2,\ 6,\ 2) + t(9,\ 2,\ 4)$ is entirely contained on the plane spanned by $\vec{u}$ and $\vec{v}$.

Question 6. [4] Let $\vec{u} = (2,\ -1,\ 1)$ and $\vec{v} = (-3,\ k,\ k^2)$

a. *(2 marks)* Find all values of k for which $\vec{u}$ and $\vec{v}$ are orthogonal.

b. *(1 mark)* Find a unit vector that is orthogonal to $\vec{u}$.

c. *(4 marks)* Find all values of k for which $\{\vec{u},\ \vec{v},\ \vec{u} \times \vec{v}\}$ is linearly independent.

[3] from a John Abbott final examination

[4] partly from a John Abbott final examination

Question 7. [5] Let $\{\vec{u},\ \vec{v},\ \vec{w}\}$ be a set of linearly independent vectors in $\mathbb{R}^3$

a. *(3 marks)* Simplify $\vec{u} \cdot [(\vec{v}-\vec{u}) \times (\vec{w}-\vec{u})]$.

b. *(2 marks)* Prove or disprove: The parallelepiped with sides $\vec{u}$, $\vec{v}$ and $\vec{w}$ has the same volume as the parallelepiped with sides $\vec{u}$, $\vec{v}-\vec{u}$ and $\vec{w}-\vec{u}$.

Question 8. The number of leading 1's in a row echelon form of A is called the *rank* of A. Let $V = \{M \mid M \in \mathscr{M}_{n\times n} \text{ and } \operatorname{rank}(M) < n\}$ with vector addition defined as matrix multiplication and scalar multiplication defined as $k \cdot M = kM + rI - I$.

a. *(3 marks)* Is the zero vector an element of V? Justify.

b. *(2 marks)* Determine whether the following axiom holds: $(rs)M = r(sM)$ where $r,\ s \in \mathbb{R}$ and $M \in V$.

c. *(1 mark)* Is V with the given operations a vector space, Justify.

[5] from a John Abbott final examination

Question 9. *(3 marks)* The *union* of two sets U and W is defined as $U \bigcup W = \{x \mid x \in U \text{ or } x \in W\}$. Prove or disprove: The union of any two subspaces of a vector space V is a subspace of V.

Question 10. Given $W = \left\{a + bx + cx^2 + dx^3 \mid a + 2b + 3c + 4d = 0 \text{ and } b + c + d = 0\right\}$ a subspace of $\mathscr{P}_3$.

a. *(4 marks)* Find a basis B for W.

b. *(1 mark)* State the $\dim(W)$ and $\dim(\mathscr{P}_3)$.

c. *(2 marks)* Express $p(x) = -2 - 3x + 4x^2 - x^3$ relative to the basis found in part a.

Bonus. *(3 marks)* Prove the commutativity axiom, assuming the other nine vector space axioms.