Dawson College: **Linear Algebra**: 201-NYC-05-S02: Winter 2018

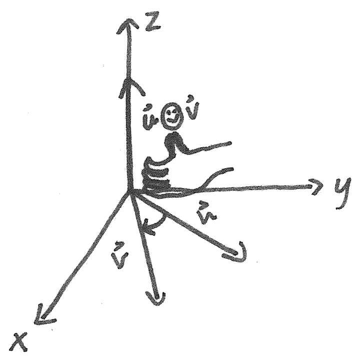
Test 2

This test is graded out of 48 marks. No books, notes, watches or cell phones are allowed. You are only permitted to use the Sharp EL-531 calculator. Give the work in full unless otherwise stated, reduce each answer to its simplest exact form. Write and arrange your exercise in a legible and orderly manner. If you need more space for your answer use the back of the page.

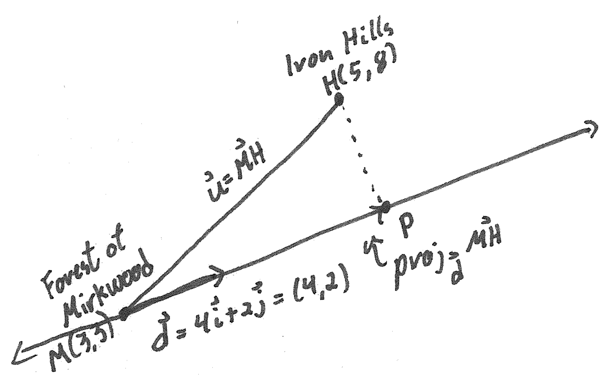
Question 1. *(2 marks)* After years of studying mathematics Y. Lamontagne *(<- write your name)* defined a vector product in $\mathbb{R}^3$ which has the same magnitude as the cross product but the direction of the product is given by the left-hand rule. The name of the product is the *happy product* denoted by ☺ and defined as

$$\vec{u} ☺ \vec{v} = (u_1, u_2, u_3) ☺ (v_1, v_2, v_3) = \left(-\begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \right)$$ *(<- write the correct formula for the happy product)*

Draw a sketch illustrating the left-hand rule and the happy product of two vectors.



Question 2.[1] *(5 marks)* Gandalf the Grey started in the Forest of Mirkwood at a point with coordinates (3, 3) and arrived in the Iron Hills at the point with coordinates (5, 8). If he began walking in the direction of the vector $\vec{v} = 4\vec{i} + 2\vec{j}$ and changes direction only once, when he turns at a right angle, what are the coordinates of the point where he makes the turn.



$\vec{MH} = H - M = (5,8) - (3,3) = (2,5)$

$$\text{proj}_{\vec{d}} \vec{MH} = \frac{\vec{d} \cdot \vec{MH}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{(4,2)\cdot(2,5)}{(4,2)\cdot(4,2)}(4,2) = \frac{18}{20}(4,2) = \frac{9}{10}(4,2) = \frac{9}{5}(2,1)$$

$$\vec{MP} = \text{proj}_{\vec{d}} \vec{MH}$$

$$P - M = \frac{9}{5}(2,1)$$

$$P = M + \frac{9}{5}(2,1) = (3,3) + \frac{9}{5}(2,1) = \left(\frac{33}{5}, \frac{24}{5}\right)$$

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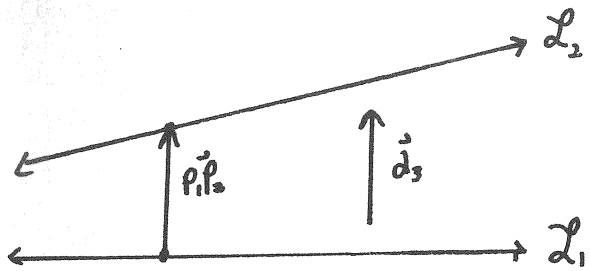
[1] from WeBWorK

Question 3.[2] *(5 marks)* Given the lines:

$$\begin{aligned}
\mathscr{L}_1 &: (x,y,z) = (1,2,-2) + t_1(1,2,1)\\
\mathscr{L}_2 &: (x,y,z) = (2,1,3) + t_2(1,2,3)\\
\mathscr{L}_3 &: (x,y,z) = (1,1,1) + t_3(2,7,3) \text{ where } t_1,\ t_2,\ t_3 \in \mathbb{R}.
\end{aligned}$$

Find the equation of the line which is parallel to $\mathscr{L}_3$ and which intersects both $\mathscr{L}_1$ and $\mathscr{L}_2$.



To find the equation of the line we can find the vector from $\mathscr{L}_1$ to $\mathscr{L}_2$ and parallel to $\vec{d_3}$.

$$\vec{P_1P_2} = (2+t_2,\ 1+2t_2,\ 3+3t_2) - (1+t_1,\ 2+2t_1,\ -2+t_1)$$
$$= (1+t_2-t_1,\ -1+2t_2-2t_1,\ 5+3t_2-t_1)$$

$$\vec{P_1P_2} = k\vec{d_3}$$

$$(1+t_2-t_1,\ -1+2t_2-2t_1,\ 5+3t_2-t_1) = k(2,7,3)$$

$$\left.\begin{aligned} 1+t_2-t_1 &= 2k\\ -1+2t_2-2t_1 &= 7k\\ 5+3t_2-t_1 &= 3k \end{aligned}\right\} \Rightarrow \begin{aligned} t_2-t_1-2k &= -1\\ 2t_2-2t_1-7k &= 1\\ 3t_2-t_1-3k &= -5 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -1\\ 2 & -2 & -7 & 1\\ 3 & -1 & -3 & -5 \end{array}\right]$$

$$\sim \begin{array}{l} \\ -2R_1+R_2 \to R_2\\ -3R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 1 & -1 & -2 & -1\\ 0 & 0 & -3 & 3\\ 0 & 2 & 3 & -2 \end{bmatrix}$$

$$\sim R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & -1 & -2 & -1\\ 0 & 2 & 3 & -2\\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$$\sim -\tfrac{1}{3}R_3 \to R_3 \begin{bmatrix} 1 & -1 & -2 & -1\\ 0 & 2 & 3 & -2\\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{array}{l} 2R_3+R_1 \to R_1\\ -3R_3+R_2 \to R_2 \end{array} \begin{bmatrix} 1 & -1 & 0 & -3\\ 0 & 2 & 0 & 1\\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \tfrac{1}{2}R_2 \to R_2 \begin{bmatrix} 1 & -1 & 0 & -3\\ 0 & 1 & 0 & \tfrac{1}{2}\\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim R_2+R_1 \to R_1 \begin{bmatrix} 1 & 0 & 0 & -\tfrac{5}{2}\\ 0 & 1 & 0 & \tfrac{1}{2}\\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\therefore$ point on $\mathscr{L}_1$ when $t_1 = \tfrac{1}{2}$

$$(x,y,z) = (1,2,-2) + \tfrac{1}{2}(1,2,1)$$
$$= \left(\tfrac{3}{2},\ 3,\ \tfrac{-3}{2}\right)$$

$$\therefore \mathscr{L}: (x,y,z) = \left(\tfrac{3}{2},\ 3,\ \tfrac{-3}{2}\right) + t(2,7,3)$$

[2] from Winter 2017 Final Examination

Question 4. *(5 marks)* Determine whether the two lines intersect, are parallel or are skew lines. Find the shortest distance between the lines using projections.

$$\mathcal{L}_1 : \begin{cases} x = 3t \\ y = -1+2t \\ z = 1+2t \end{cases} \text{ and } \mathcal{L}_2 : \begin{cases} x = 1-6t \\ y = -4t \\ z = 2-4t \end{cases}$$

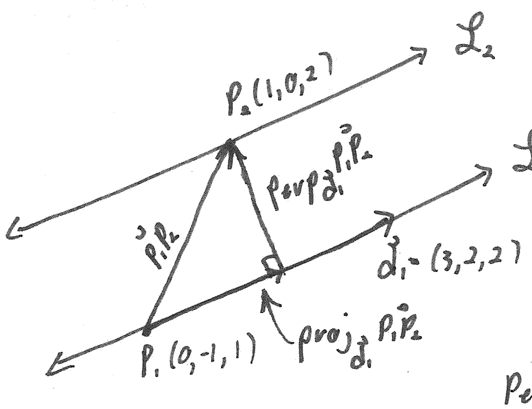
$\mathcal{L}_1 : \vec{x} = (0,-1,1) + t(3,2,2)$

$\mathcal{L}_2 : \vec{x} = (1,0,2) + t(-6,-4,-4)$

$\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel since $\vec{d}_1 = (3,2,2)$ is a multiple of $\vec{d}_2 = (-6,-4,-4)$. That is the direction vectors are parallel to each other. $\mathcal{L}_1$ and $\mathcal{L}_2$ are not identical lines since $P_1(0,-1,1)$ does not lie on $\mathcal{L}_2$:

$(0,-1,1) = (1,0,2) + t(-6,-4,-4)$

① $0 = 1-6t$
② $-1 = -4t$ $\Rightarrow t = 1/4$ which does not satisfy ①.
③ $1 = 2-4t$



$\vec{P_1P_2} = P_2 - P_1 = (1,0,2) - (0,-1,1) = (1,1,1)$

$$\text{proj}_{\vec{d}_1} \vec{P_1P_2} = \frac{\vec{d}_1 \cdot \vec{P_1P_2}}{\vec{d}_1 \cdot \vec{d}_1} \vec{d}_1$$

$$= \frac{(3,2,2)\cdot(1,1,1)}{(3,2,2)\cdot(3,2,2)}(3,2,2) = \frac{7}{17}(3,2,2)$$

$$\text{Perp}_{\vec{d}_1} \vec{P_1P_2} = \vec{P_1P_2} - \text{proj}_{\vec{d}_1} \vec{P_1P_2}$$

$$= (1,1,1) - \frac{7}{17}(3,2,2)$$

$$= \left(\frac{-4}{17}, \frac{3}{17}, \frac{3}{17}\right)$$

distance $= \|\text{Perp}_{\vec{d}_1} \vec{P_1P_2}\|$

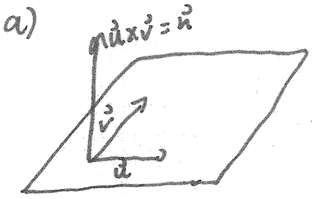
$$= \sqrt{\left(\frac{-4}{17}\right)^2 + \left(\frac{3}{17}\right)^2 + \left(\frac{3}{17}\right)^2} = \frac{1}{17}\sqrt{(-4)^2+3^2+3^2} = \frac{\sqrt{34}}{17}$$

Question 4. [3] Let $\vec{u} = (1,\ 3,\ 1)$ and $\vec{v} = (2,\ 1,\ 1)$.

a. *(3 marks)* Find an equation of the form $ax_1 + bx_2 + cx_3 = d$ for the plane spanned by $\vec{u}$ and $\vec{v}$.

b. *(2 marks)* Show that the line $(x_1,\ x_2,\ x_3) = (2,\ 6,\ 2) + t(9,\ 2,\ 4)$ is entirely contained on the plane spanned by $\vec{u}$ and $\vec{v}$.

a)



$\vec{n} = \vec{u}\times\vec{v} = \left(\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\right) = (2,\ 1,\ -5)$

$\begin{matrix} 1 & 2 \\ 3 & 1 \\ 1 & 1 \end{matrix}$

$2x_1 + x_2 - 5x_3 = 0$

$d = 0$ since the plane passes through the origin.

b) $\vec{d}\cdot\vec{n} = (9,2,4)\cdot(2,1,-5) = 0$

$\therefore$ the line and plane are parallel.

Does $(2,6,2)$ lie on the plane?

$2(2) + 6 - 5(2) = 0$ yes.

$\therefore$ the line is completely contained by the plane.

Question 5. [4] Let $\vec{u} = (2,\!-1,\ 1)$ and $\vec{v} = (-3,\ k,\ k^2)$

a. *(2 marks)* Find all values of k for which $\vec{u}$ and $\vec{v}$ are orthogonal.

b. *(1 mark)* Find a unit vector that is orthogonal to $\vec{u}$.

c. *(3 marks)* Find all values of k for which $\{\vec{u},\ \vec{v},\ \vec{u}\times\vec{v}\}$ is linearly independent.

a) $\vec{u}\cdot\vec{v} = 0$

$0 = (2,-1,1)\cdot(-3,k,k^2)$

$0 = -2(3) - k + k^2$

$0 = -6 - k + k^2$

$0 = (k+2)(k-3)$

$k = -2 \qquad k = 3$

$\therefore$ orthogonal when $k=-2$ and $k=3$.

b) if $k=-2$ then $\vec{v} = (-3,-2,4)$ is orthogonal to $\vec{u}$. The unit vector in the same direction as $\vec{v}$ is

$\dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{(-3)^2+(-2)^2+4^2}}(-3,-2,4) = \dfrac{1}{\sqrt{29}}(-3,-2,4)$

c) Note that there does not exist a l s.t. $\vec{u} = l\vec{v}$ for any k. ♠

$c_1\vec{u} + c_2\vec{v} + c_3(\vec{u}\times\vec{v}) = \vec{0}$

Taking the dot product with $\vec{u}\times\vec{v}$

$(c_1\vec{u} + c_2\vec{v} + c_3(\vec{u}\times\vec{v}))\cdot(\vec{u}\times\vec{v}) = \vec{0}\cdot(\vec{u}\times\vec{v})$

$c_1\underbrace{(\vec{u}\cdot(\vec{u}\times\vec{v}))}_{0} + c_2\underbrace{(\vec{v}\cdot(\vec{u}\times\vec{v}))}_{0} + c_3((\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{v})) = 0$

$c_3\|\vec{u}\times\vec{v}\|^2 = 0$

$c_3 = 0$ since $\|\vec{u}\times\vec{v}\| \neq 0$ by ♠

Since $c_3 = 0$, we have

$c_1\vec{u} + c_2\vec{v} = 0$

which implies $c_1 = c_2 = 0$ or else it would contradict ♠.

$\therefore c_1 = c_2 = c_3 = 0$ (only the trivial solution)

$\therefore$ the set is linearly independent.

[3] from a John Abbott final examination

[4] partly from a John Abbott final examination

Question 6. [5] Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be a set of linearly independent vectors in $\mathbb{R}^3$

a. *(3 marks)* Simplify $\vec{u} \cdot [(\vec{v} - \vec{u}) \times (\vec{w} - \vec{u})]$.

b. *(2 marks)* Prove or disprove: The parallelepiped with sides $\vec{u}$, $\vec{v}$ and $\vec{w}$ has the same volume as the parallelepiped with sides $\vec{u}$, $\vec{v} - \vec{u}$ and $\vec{w} - \vec{u}$.

a) $\vec{u} \cdot [(\vec{v} - \vec{u}) \times (\vec{w} - \vec{u})]$

$= \vec{u} \cdot [\vec{v} \times \vec{w} + (-\vec{u} \times \vec{w}) + \vec{v} \times (-\vec{u}) + (-\vec{u}) \times (-\vec{u})]$

$= \vec{u} \cdot [\vec{v} \times \vec{w} - \vec{u} \times \vec{w} - \vec{v} \times \vec{u}]$ since $\vec{u} \times \vec{u} = \vec{0}$

$= \vec{u} \cdot [\vec{v} \times \vec{w}] + \vec{u} \cdot [-(\vec{u} \times \vec{w})] + \vec{u} \cdot [-(\vec{v} \times \vec{u})]$

$= \vec{u} \cdot [\vec{v} \times \vec{w}] - \vec{u} \cdot (\vec{u} \times \vec{w}) - \vec{u} \cdot (\vec{v} \times \vec{u})$

$= \vec{u} \cdot [\vec{v} \times \vec{w}]$

since $\vec{u} \cdot (\vec{u} \times \vec{w}) = \vec{0}$

and $\vec{u} \cdot (\vec{v} \times \vec{u}) = \vec{0}$

b) Prove:

By part a)

$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{u} \cdot [(\vec{v} - \vec{u}) \times (\vec{w} - \vec{u})]$

It follows that

$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |\vec{u} \cdot [(\vec{v} - \vec{u}) \times \vec{w} - \vec{u})]|$

and as shown in class

Volume $= |\vec{u} \cdot (\vec{v} \times \vec{w})|$

hence the same volume.

Question 7. *(5 marks)* The number of leading 1's in a row echelon form of A is called the *rank* of A. Let $V = \{M \mid M \in \mathcal{M}_{n \times n}$ and $\text{rank}(M) < n\}$ with vector addition defined as matrix multiplication and scalar multiplication defined as $k \cdot M = KM + KI - I$

a. *(3 marks)* Is the zero vector an element of V? Justify.

b. *(3 marks)* Determine whether the following axiom holds: $(rs)M = r(sM)$ where $r, s \in \mathbb{R}$ and $M \in V$.

c. *(1 mark)* Is V with the given operations a vector space, Justify.

a) Let $M \in V$ and $O \in M_{n \times n}$

$M + O = M$

$MO = M$

implies that $O = I$. But

$I \notin V$ since $\text{rank}(I) = n$

b) LHS $= (rs)M = (rs)M + (rs)I - I$

RHS $= r(sM) = r(sM + sI - I) = r(sM + sI - I) + rI - I$

$= rsM + rsI - rI + rI - I$

$= rsM + rsI - I$

$=$ RHS

$\therefore$ the axiom holds.

c) V is not a vector space because of part a.

[5]from a John Abbott final examination

Question 9. *(3 marks)* The ~~intersection~~ Union of two sets U and W is defined as $U \cup W = \{x \mid x \in U \text{ or } x \in W\}$. Prove or disprove: The ~~intersection~~ Union of any two subspaces of a vector space V is a subspace of V.

disprove: Let $U = \{(x,0) \mid x \in \mathbb{R}\}$

$V = \{(0,y) \mid y \in \mathbb{R}\}$ both are subspaces of $\mathbb{R}^2$ since they are both lines that passes through the origin.

$(1,0) \in U$ and $(0,1) \in V$ but $(1,0)+(0,1) = (1,1) \notin U \cup V$. So not closed under addition. $\therefore$ not a subspace

Question 10. Given $W = \{a+bx+cx^2+dx^3 \mid a+2b+3c+4d = 0 \text{ and } b+c+d = 0\}$ a subspace of $\mathscr{P}_3$.

(① and ② are written above the two conditions.)

a. *(4 marks)* Find a basis B for W.

b. *(1 mark)* State the dim(W) and dim($\mathscr{P}_3$).

c. *(2 marks)* Express $p(x) = 2 - 3x + 4x^2 - x^3$ relative to the basis found in part a.

b) $\dim(W) = 2$ and $\dim(\mathscr{P}_3) = 4$

a) For a pol. to be part of W must satisfy both ① and ②

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \sim^{-2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Let $c = s$, $d = t$, $s,t \in \mathbb{R}$ $\quad a = -s-2t$, $b = -s-t$

$p(x) = a+bx+cx^2+dx^3 = -s-2t + (-s-t)x + sx^2 + tx^3$

$= s\underbrace{(-1-x+x^2)}_{p_1(x)} + t\underbrace{(-2-x+x^3)}_{p_2(x)}$

So $\text{span}(\{p_1(x), p_2(x)\}) = W$ and since $p_1(x)$ and $p_2(x)$ are not multiples of each other they are linearly independent.

$\therefore B = \{p_1(x), p_2(x)\}$ is a basis of W.

c) $p(x) = c_1 p_1(x) + c_2 p_2(x)$

$-2-3x+4x^2-x^3 = c_1(-1-x+x^2) + c_2(-2-x+$

$\therefore c_2 = -1$

$c_1 = 4$

$\therefore (p(x))_B = (4,-1)$

Bonus. *(3 marks)* Prove the commutativity axiom, assuming the other nine vector space axioms.

definition of vector space needs no commutativity

In the definition of vector space (http://planetmath.org/VectorSpace) one usually lists the needed properties of the vectoral addition and the multiplication of vectors by scalars as eight axioms, one of them the commutative law

$$u+v=v+u.$$

The latter is however not necessary, because it may be proved to be a consequence of the other seven axioms. The proof can be based on the fact that in defining the group (http://planetmath.org/Group), it suffices to postulate only the existence of a right identity element and the right inverses of the elements (see the article "redundancy of two-sidedness in definition of group (http://planetmath.org/RedundancyOfTwoSidednessInDefinitionOfGroup)").

Now, suppose the validity of the seven other axioms (http://planetmath.org/VectorSpace), but not necessarily the above commutative law of addition. We will show that the commutative law is in force.

We need the identity $(-1)\,v=-v$ which is easily justified (we have $\vec{0}=0v=(1+(-1))\,v=\ldots$). Then we can calculate as follows:

$$\begin{aligned}
v+u &= (v+u)+\vec{0}=(v+u)+[-(u+v)+(u+v)]\\
&= [(v+u)+(-(u+v))]+(u+v)=[(v+u)+(-1)\ (u+v)]+(u+v)\\
&= [(v+u)+((-1)\,u+(-1)\,v)]+(u+v)=[((v+u)+(-u))+(-v)]+(u+v)\\
&= [(v+(u+(-u)))+(-v)]+(u+v)=[(v+\vec{0})+(-v)]+(u+v)\\
&= [v+(-v)]+(u+v)=\vec{0}+(u+v)\\
&= u+v
\end{aligned}$$

Q.E.D.

This proof by Y. Chemiavsky and A. Mouftakhov is found in the 2012 March issue of *The American Mathematical Monthly*.

Title	definition of vector space needs no commutativity
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