Dawson College: Winter 2019: Linear Algebra (SCIENCE): 201-NYC-05-S6: **Quiz 15** name: Y. Lamontagne

No books, watches, notes or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Definitions. Let A be an $m \times n$ matrix.
The set of all $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$ is called the *null space* of A, that is, $\mathrm{Nul}(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$.
The set of all $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{x}$ is in the span of the vectors that are the columns of A is called the *column space* of A, that is, if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the columns of A then $\mathrm{Col}(A) = \mathrm{span}(\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\})$.

Question 1.[1] *(1 mark each)* Complete the following sentences with the word **must**, **might** or, **cannot**, as appropriate.

a. The columns of an $n \times n$ elementary matrix must be a basis for $\mathbb{R}^n$.

b. If A is an $m \times n$ matrix and $\dim(\mathrm{Nul}(A)) = n$, then must A be a $m \times n$ zero matrix

Question 2. *(1 mark each)* Fill in the blank.

a. The vector space of all diagonal $n \times n$ matrices has dimension n.

b. The vector space of all skew-symmetric (i.e. $A^T = -A$) $n \times n$ matrices has dimension $\frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$

Question 3.[1] Consider the following matrix A and its reduced row echelon form B given below. Justify completely!

$$A = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{matrix} \\ \begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 3 \\ 3 & 4 & -14 & 1 & 0 & 14 \\ 1 & -1 & 7 & 2 & 0 & -9 \\ -3 & -2 & 4 & 3 & 0 & -24 \end{bmatrix} \end{array} \qquad B = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & -5 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. *(4 marks)* Find a basis for $\mathrm{Col}(A)$.

$\mathcal{B} = \{\underline{a}_1, \underline{a}_2, \underline{a}_4\}$ is linearly independent since it follows from the RREF (B) of A that $c_1\underline{a}_1 + c_2\underline{a}_2 + c_3\underline{a}_4 = \underline{0}$ has only the trivial solution.
Is $\mathrm{span}(\mathcal{B}) = \mathrm{span}(\{\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5, \underline{a}_6\})$? $\underline{a}_5 \in \mathrm{span}(\mathcal{B})$ since $\underline{0} = 0\cdot\underline{a}_1 + 0\underline{a}_2 + 0\cdot\underline{a}_4$, $\underline{a}_3 \in \mathrm{span}(\mathcal{B})$ since $\underline{a}_3 = 2\underline{a}_1 - 5\underline{a}_2 + 0\underline{a}_4$*. $\underline{a}_6 \in \mathrm{span}(\mathcal{B})$ since $\underline{a}_6 = 2\underline{a}_1 + 3\underline{a}_2 - 4\underline{a}_4$. $\therefore \mathrm{span}(\mathcal{B}) = \mathrm{span}(\{\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4, \underline{a}_5, \underline{a}_6\})$

b. *(4 marks)* Find a basis for $\mathrm{Nul}(A)$.

$$[A|0] \sim \text{Gauss Jordan} \sim [B|0] = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & -5 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $x_3 = s$, $x_5 = t$, $x_6 = u$, $s, t, u \in \mathbb{R}$

$\therefore (x_1, x_2, x_3, x_4, x_5, x_6)$
$= (-2u-2s, -3u+5s, s, 4u, t, u)$
$= u\underbrace{(-2,-3,0,4,0,0)}_{\underline{b}_1} + s\underbrace{(-2,5,1,0,0,0)}_{\underline{b}_2} + t\underbrace{(0,0,0,0,1,0)}_{\underline{b}_3}$

$\therefore \mathrm{Nul}(A) = \mathrm{span}(\{\underline{b}_1, \underline{b}_2, \underline{b}_3\})$

$S = \{\underline{b}_1, \underline{b}_2\}$ is linearly indep. since the vectors are not multiples of each other. In addition $\underline{b}_3 \notin \mathrm{span}(S)$ therefore $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \underline{b}_3\}$ is linearly independent.
$\therefore \mathcal{B}$ is a basis for $\mathrm{Nul}(A)$

c. *(1 mark)* Express the third column of matrix A relative to the basis found in part a.

$(\underline{a}_3)_{\mathcal{B}} = \mathrm{Rep}_{\mathcal{B}}(\underline{a}_3) = (2, -5, 0)$ by *

d. *(1 mark)* Determine the dimension of $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$.

$\dim(\mathrm{Col}(A)) = 3$ and $\dim(\mathrm{Nul}(A)) = 3$

[1] From or modified from John Abbott College.