Dawson College: Winter 2020: Linear Algebra (SCIENCE): 201-NYC-05-S3: **Test 1, part 1 of 2** name: ____________________

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.[1] *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. Let A be a square matrix. If $A\mathbf{x} = A\mathbf{y}$ for distinct $\mathbf{x}$ and $\mathbf{y}$, then A ______________ be invertible.

b. If the 3×3 coefficient matrix A has a RREF with two leading ones, then the system $A\mathbf{x} = \mathbf{b}$ ______________ be inconsistent when $\mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

c. If A, B,and C are square matrices such that $ABC^2 = I$ and A is invertible then matrix B ______________ be invertible.

d. If A is a square matrix and $A^4 - 2A^2 + A = I$, then A ______________ be invertible.

e. If A and B are $n \times n$ matrices such that $AB = B$, then A ______________ be an identity matrix.

f. Given an $n \times n$ matrix A. If the system $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then the system $A\mathbf{x} = \mathbf{0}$ ______________ have non-trivial solutions.

g. If E_1 and E_2 are two elementary matrices, then E_1E_2 ______________ be equal to E_2E_1.

h. The expression $(I - A)(I + A)$ ______________ be equal to $I - A^2$.

i. For any invertible matrix A, the number of leading ones of the RREF of A ______________ be the same as the number of leading ones of the RREF of A^2.

j. If E_1, E_2 are elementary matrices, then E_1E_2 ______________ also be an elementary matrix.

k. If matrix AB is invertible, then A ______________ be invertible

Question 2.[1] *(2 marks)* Suppose $\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Show that $\mathbf{w} = 3\mathbf{u} - 4\mathbf{v}$ is a solution to $A\mathbf{x} = 3\mathbf{b}$.

Question 3. *(1.5 marks)* Illustrate **all** relative positions of lines in an inconsistent linear system of three lines.

Question 4. *(4 marks)* Find **all** 2×2 symmetric matrices A such that $A \begin{bmatrix} 1 & -2 \end{bmatrix}^T - \begin{bmatrix} 3 & 4 \end{bmatrix}^T = \mathbf{0}$

[1] From or modified from a John Abbott final examination

Question 5.[1] *(5 marks)* Let $A = \begin{bmatrix} 1 & 1 & a \\ 1 & a & a \\ a & a & a \\ a & a & a^2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a^2 - 2a \end{bmatrix}$. Find the value(s) of a, if possible, for which the equation $A\mathbf{x} = \mathbf{b}$ has:

a. a unique solution,

b. infinitely many solutions,

c. no solution.

Question 6. *(5 marks)* Consider the matrix equation $E_1^{-1}E_2 = (E_3 - \text{tr}(I_5)A)^{-1}$ where $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Solve for A.

Question 7. *(5 marks)* Find elementary matrices E_1, E_2 and E_3 which satisfy the following equation: $E_3E_2E_1 \begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = I$

Question 8. *(3 marks each)* A square matrix A is *idempotent* if $A^2 = A$.

a. Prove that if A and B are idempotent and are commutative, then AB is idempotent.

b. Prove that if A is idempotent then either A is singular or $A = I$. *Hint: Prove by contradiction.*

Question 9. *(5 marks)* Given A an $n \times n$ invertible matrix. Show that the reduced row echelon form B is I_n if and only if $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Bonus Question. *(5 marks)* Enumerate all the solution(s) of $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

given that the numbers are $\mathbb{Z}_3$ instead of $\mathbb{R}$. Operations on the numbers of $\mathbb{Z}_3$ can be defined by the following Cayley tables:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1