Dawson College: Winter 2020: Linear Algebra (SCIENCE): 201-NYC-05-S3: **Test 1** name: Y. Lamontagne

No books, watches, notes or cell phones are allowed. The **only** calculators allowed are the Sharp EL-531. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.[1] *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. Let A be a square matrix. If $A\mathbf{x} = A\mathbf{y}$ for distinct $\mathbf{x}$ and $\mathbf{y}$, then A __cannot__ be invertible.

b. If the 3×3 coefficient matrix A has rank 2, then the system $A\mathbf{x} = \mathbf{b}$ __might__ be inconsistent when $\mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

c. If A, B,and C are square matrices such that $ABC^2 = I$ and A is invertible then matrix B __must__ be invertible.

d. If A is a square matrix and $A^4 - 2A^2 + A = I$, then A __must__ be invertible.

e. If A and B are $n \times n$ matrices such that $AB = B$, then A __might__ be an identity matrix.

f. Given an $n \times n$ matrix A. If the system $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then the system $A\mathbf{x} = \mathbf{0}$ __must__ have non-trivial solutions.

g. If E_1 and E_2 are two elementary matrices, then E_1E_2 __might__ be equal to E_2E_1.

h. The expression $(I - A)(I + A)$ __must__ be equal to $I - A^2$.

i. For any invertible matrix A, the rank of A __must__ be the same as the rank of A^2.

j. If E_1, E_2 are elementary matrices, then E_1E_2 __might__ also be an elementary matrix.

k. If matrix AB is invertible, then A __might__ be invertible

Question 2.[1] *(2 marks)* Suppose $\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Show that $\mathbf{w} = 3\mathbf{u} - 4\mathbf{v}$ is a solution to $A\mathbf{x} = 3\mathbf{b}$.

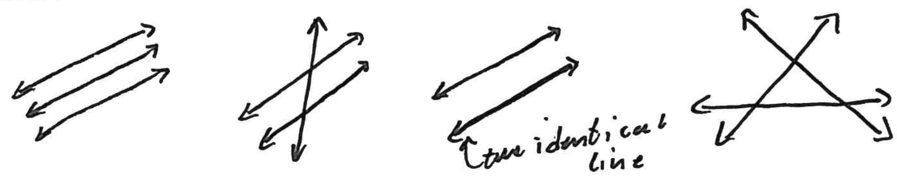
Lets show that $\underline{w}$ satisfies $A\underline{x} = 3\underline{b}$

$$
\begin{aligned}
A\underline{w} = A(3\underline{u} - 4\underline{v}) &= A(3\underline{u}) - A(4\underline{v}) \\
&= 3A\underline{u} - 4A\underline{v} \\
&= 3\underline{b} - 4\underline{0} \quad \text{since } A\underline{u} = \underline{b} \text{ and } A\underline{v} = \underline{0} \\
&= 3\underline{b}
\end{aligned}
$$

Question 3. *(2 marks)* ~~Discuss~~ Illustrate all ~~the~~ relative positions of lines ~~of~~ in an inconsistent linear system consisting of three lines.

Possible cases:

Case 1: Case 2: Case 3: Case 4:



Question 4. *(4 marks)* Find **all** 2×2 symmetric matrices A such that $A\begin{bmatrix} 1 & -2 \end{bmatrix}^T - \begin{bmatrix} 3 & 4 \end{bmatrix}^T = \mathbf{0}$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ since A is symmetric $A^T = A$ so $c = b$.

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a - 2b \\ b - 2d \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$a - 2b = 3$

$b - 2d = 4$

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -2 & 4 \end{bmatrix}$$

$$\sim 2R_2 + R_1 \to R_1 \begin{bmatrix} 1 & 0 & -4 & 11 \\ 0 & 1 & -2 & 4 \end{bmatrix}$$

Let $d = t$, $t \in \mathbb{R}$

$a = 11 + 4t$

$b = 4 + 2t$

$$\therefore A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 11 + 4t & 4 + 2t \\ 4 + 2t & t \end{bmatrix}$$

where $t \in \mathbb{R}$

[1] From or modified from a John Abbott final examination

Question 5.[1] *(5 marks)* Let $A = \begin{bmatrix} 1 & 1 & a \\ 1 & a & a \\ a & a & a \\ a & a & a^2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a^2 - 2a \end{bmatrix}$. Find the value(s) of a, if possible, for which the equation $A\mathbf{x} = \mathbf{b}$ has:

a. a unique solution,

b. infinitely many solutions,

c. no solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 1 & a & a & 0 \\ a & a & a & 0 \\ a & a & a^2 & a^2-2a \end{array}\right] \sim \begin{array}{c} -R_1+R_2 \to R_2 \\ -aR_1+R_3 \to R_3 \\ -aR_1+R_4 \to R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 0 & -1 \\ 0 & 0 & a-a^2 & -a \\ 0 & 0 & 0 & a^2-3a \end{array}\right]$$

Lets look at all case where leading entries vanish:

Case 1: $a^2 - 3a = 0$

$a(a-3) = 0$

$a = 0$ or $a = 3$

If $a = 0$ then the aug. mat becomes

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So ∞ many sol. since no leading entry in constant column and # leading entry < # var

If $a = 3$ then the aug. mat. becomes

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So unique solution since no leading entry in constant column and # leading entry = # var.

Case 2: $a - a^2 = 0$

$a(1-a) = 0$

$a = 0$ or $a = 1$

Already analysed $a = 0$

If $a = 1$ then the aug. mat. becomes

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

leading entry in constant column $\therefore$ no solution

Case 3: $a - 1 = 0$

$a = 1$

already analysed

Lets look at the cases when the leading entries do not vanish

Case 4: $a^2 - 3a \neq 0$

$a \neq 0$ and $a \neq 3$

then no solution since leading entry in constant column.

all cases have been covered.

Question 6. *(5 marks)* Consider the matrix equation $(E_1)^{-1}E_2 = (E_3 - \mathrm{tr}(\underset{5}{I})A)^{-1}$ where $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Solve for A. Note: E_1, E_2, E_3 are elementary matrices

$$E_1^{-1}E_2 = (E_3 - 5A)^{-1}$$

$$(E_1^{-1}E_2)^{-1} = (E_3 - 5A)$$

$$5A = E_3 - E_2^{-1}E_1$$

$$A = \frac{1}{5}(E_3 - E_2^{-1}E_1)$$

$$= \frac{1}{5}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{5}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1/3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{5}\left(\begin{bmatrix} 1 & -1/3 & 0 \\ -1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1/5 & -1/15 & 0 \\ -1/5 & 1/5 & 2/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 7. *(5 marks)* Find elementary matrices E_1, E_2 and E_3 which satisfy the following equation: $E_3E_2E_1 \begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = I_n$

$$\begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim_{-R_2 + R_3 \to R_3} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim_{\frac{1}{5}R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E_1:

$$I_3 \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

E_2:

$$I_3 \sim_{-R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = E_2$$

E_3:

$$I_3 \sim_{\frac{1}{5}R_1 \to R_1} \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

Question 8. *(3 marks each)* A square matrix A is *idempotent* if $A^2 = A$.

a. Prove that if A and B are idempotent and are commutative, then AB is idempotent.

b. Prove that if A is idempotent then either A is singular or $A = I$. *Hint: Prove by contradiction.*

a) Premise:

A and B are idempotent, so $A^2 = A$ and $B^2 = B$

A and B are commutative, so $AB = BA$

Conclusion:

$(AB)^2 = AB$

$LHS = (AB)^2$

$= (AB)AB$

$= AABB$ since they are commutative

$= A^2B^2$

$= AB$ since they are both idempotent

b) Premise:

A is idempotent

Conclusion:

A is singular or $A = I$

Suppose A is not singular and $A \neq I$. So A is invertible. By the premise $A^2 = A$

$A^{-1}AA = A^{-1}A$

$IA = I$

$A = I$ ↯ $\therefore$ A is singular or $A = I$

Question 9. *(5 marks)* Given A an $n \times n$ invertible matrix. Show that the reduced row echelon form B is I_n if and only if $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

[$\Rightarrow$] premise:

- A is invertible
- RREF of B is I

conclusion:

$(AB)\underline{x} = \underline{0}$ has only the trivial solution.

Since the RREF of B is I, B is invertible by the equivalence theorem. As seen in class the product of two invertible matrix is inv. Hence AB is inv. so by the equivalence thm $(AB)x = 0$ has only the trivial solution.

[$\Leftarrow$] premise:

- A is invertible
- $(AB)\underline{x} = \underline{0}$ has only the trivial solution

conclusion:

- RREF of B is I_n

Since AB has only the trivial solution then it is invertible by the equivalence thm. So

$$(AB)^{-1}AB = I$$

which implies that B is invertible and its inverse is $(AB)^{-1}A$. And by the equivalence thm the RREF of B is I_n.

Bonus Question. *(5 marks)* Enumerate all the solution(s) of $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

given that the numbers are $\mathbb{Z}_3$ instead of $\mathbb{R}$. Operations on the numbers of $\mathbb{Z}_3$ can be defined by the following Cayley tables:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1